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Real-Variable Characterizations and Its Applications of Anisotropic Ball Campanato and Hardy Spaces

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各向异性球 Campanato 空间和 Hardy 空间的实变特征及其应用

摘　要

函数空间理论及其应用是调和分析中的核心研究内容之一, 在微分方程、几何分析等学科中起着重要作用. 特别地, 将调和分析中出现的经典函数空间从 $\mathbb{R}^n$ 推广到各向异性欧氏空间被广泛应用于小波分析和偏微分方程等学科分支. 本学位论文致力于研究各向异性球 Campanato 空间和 Hardy 空间的实变特征及其应用.

设 A 是扩张矩阵, X 是 $\mathbb{R}^n$ 上的球拟 Banach 函数空间, 并假定其特定凸化满足 Fefferman–Stein 向量值不等式, 且 Hardy–Littlewood 极大算子在其另一个特定凸化上有界. 本学位论文的研究内容主要分为三个方面, 首先引入了一些与 A 和 X 相关的各向异性球 Campanato 函数空间, 并证明了这些空间是与 A 和 X 相关的各向异性 Hardy 空间 $H_X^A(\mathbb{R}^n)$ 的对偶空间. 进一步, 本学位论文首次建立了 $H_X^A(\mathbb{R}^n)$ 的各向异性 Littlewood–Paley 函数特征, 证明了 $H_X^A(\mathbb{R}^n)$ 上的某个函数 f 的 Fourier 变换在分布意义下与 $\mathbb{R}^n$ 上的某个连续函数 F 一致, 并给出了一个 F 的点态估计, 即函数 F 被 f 的各向异性 Hardy 空间范数与一个相关于扩张矩阵 A 的阶梯函数的乘积所控制. 结合上述 $H_X^A(\mathbb{R}^n)$ 上函数的 Fourier 变换结果, 本学位论文建立了 $H_X^A(\mathbb{R}^n)$ 上的 Hardy–Littlewood 不等式. 此外, 本学位论文建立了各向异性球 Campanato 函数空间的各种范数等价特征, 结合与 A 和 X 相关的各向异性帐篷空间的原子分解, 得到了各向异性球 Campanato 函数空间的 Carleson 测度特征. 本学位论文的关键创新之处在于, 将考虑的函数空间 X 嵌入到具有特定权重的各向异性加权 Lebesgue 空间中, 然后充分利用该加权 Lebesgue 空间的已知结果克服了由拟范数 $\|\cdot\|_X$ 的显式表达和绝对连续性缺失所产生的本质困难.

本学位论文的所有结果具有广泛的一般性, 特别是当它们应用于 Morrey 空间、Orlicz-slice 空间、Lorentz 空间和 Orlicz 空间时, 其中的一些结果是全新的. 当应用于经典 (变指标与混合范数)Lebesgue 空间时, 所获结果或者本质地改进了已有结果, 或者与已知的最好结果一致. 这些结果为研究偏微分方程和 (应用) 调和分析等学科提供了更多的工作空间和理论工具.

关键词: 各向异性欧氏空间, 扩张矩阵, 球拟 Banach 函数空间, Hardy 空间, Campanato 函数空间, 对偶理论, Fourier 变换.

Real-Variable Characterizations and Its Applications of Anisotropic Ball Campanato-Type and Hardy-Type Spaces

ABSTRACT

The theory of function spaces and its applications is one of the central research topics in harmonic analysis, playing a crucial role in disciplines such as differential equations and geometric analysis. Particularly, the extension of classical function spaces appearing in harmonic analysis from $\mathbb{R}^n$ to anisotropic Euclidean spaces has found widespread application in branches like wavelet analysis and partial differential equations. This dissertation is dedicated to studying the real-variable properties and applications of anisotropic ball Campanato spaces and Hardy spaces.

Let A be an expansive matrix, and X be a quasi-Banach function space on $\mathbb{R}^n$, assuming its specific convexification satisfies the Fefferman–Stein vector-valued inequality, and the Hardy–Littlewood maximal operator is bounded on another specific convexification. The research content of this dissertation is mainly divided into three aspects. Firstly, some anisotropic ball Campanato function spaces related to A and X are introduced, and it is proved that these spaces are the dual spaces of anisotropic Hardy spaces $H_X^A(\mathbb{R}^n)$ associated with A and X. Furthermore, this dissertation establishes for the first time the anisotropic Littlewood–Paley function characterizations of $H_X^A(\mathbb{R}^n)$, proving that the Fourier transform of a function f on $H_X^A(\mathbb{R}^n)$ is uniformly continuous in distribution sense with a continuous function F on $\mathbb{R}^n$, and provides a pointwise estimate of F, controlled by the norm of f in the anisotropic Hardy space and a product related to an extension matrix A of a step function. Combining the results of the Fourier transform of functions on $H_X^A(\mathbb{R}^n)$, this dissertation establishes the Hardy–Littlewood inequality on $H_X^A(\mathbb{R}^n)$. Additionally, various norm equivalent characterizations of anisotropic ball Campanato function spaces are established, and combined with the atomic decomposition of anisotropic tent spaces related to A and X, the Carleson measure characterizations of anisotropic ball Campanato function spaces are obtained. The key innovation of this dissertation is, embedding the considered function space X into anisotropic weighted Lebesgue spaces with specific weights and fully utilizing the known results of this weighted Lebesgue space to overcome the inherent difficulties arising from the explicit expression of the quasi-norm $\|\cdot\|_X$ and the lack of absolute continuity.

All the results of this dissertation have extensive generality, especially when applied to Morrey spaces, Orlicz-slice spaces, Lorentz spaces, and Orlicz spaces, some of the results are entirely new. When applied to classical (variable and mixed-norm) Lebesgue spaces, the obtained results either fundamentally improve existing results or are consistent with the best known results. These results provide more working space and theoretical tools for studying partial differential equations and (applied) harmonic analysis.

KEY WORDS: Expansive matrix, Ball quasi-Banach function space, Hardy spaces, Campanato-type function spaces, Duality, Fourier transform.

Contents

Chapter 1

Introduction

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1.1 Background and a Short Summary

c1s1

Recall that the dual theory of classical Hardy spaces on the Euclidean space $\mathbb{R}^n$ plays an important role in many branches of analysis, such as harmonic analysis and partial differential equations, and has been systematically considered and developed so far; see, for instance, [34, 79]. Indeed, in 1969, Duren et al. [32] first identified the Lipshitz space with the dual space of the Hardy space $H^p(\mathbb{D})$ of holomorphic functions, where $p \in (0,1)$ and the *symbol* $\mathbb{D}$ denotes the unit disc of $\mathbb{C}$. Later, Walsh [94] further extended this dual result to the Hardy space on the upper half-plane $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$. On the real Hardy spaces, the famous dual theorem, that is, the space $\mathrm{BMO}(\mathbb{R}^n)$ of functions with bounded mean oscillation is the dual space of the Hardy space $H^1(\mathbb{R}^n)$, is due to Fefferman and Stein [34]. Moreover, it is worth pointing out that the complete dual theory of the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ was given by Taibleson and Weiss [88], in which the dual space of $H^p(\mathbb{R}^n)$ proves the special Campanato space introduced by Campanato [16]. Also, in 1972, Fefferman and Stein [34] introduced a famous problem, that is, what is the characterization of the Fourier transform $\widehat{f}$ of a distribution f from the classical Hardy space $H^p(\mathbb{R}^n)$. Recall that, in 1974, Coifman [23] characterized $\widehat{f}$ via the entire function of exponential type for $n = 1$, where $f \in H^p(\mathbb{R})$ with $p \in (0,1]$. Since then, many researchers investigated the characterization of $\widehat{f}$ with the distribution f from Hardy spaces with $n \geq 2$; see, for instance, [9, 27, 37, 88]. In particular, Taibleson and Weiss [88] proved that, for any given $p \in (0,1]$, the Fourier transform of $f \in H^p(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions and there exists a positive constant C, independent of f and F, such that, for any $x \in \mathbb{R}^n$,

eqabs1
$$|F(x)| \leq C\|f\|_{H^p(\mathbb{R}^n)}|x|^{n(\frac{1}{p}-1)}. \tag{1.1.1}$$

This further implies the following generalization of the Hardy–Littlewood inequality that

eqabs2 (1.1.2)
$$\left[\int_{\mathbb{R}^n} |x|^{n(p-2)}|F(x)|^p\,dx\right]^{1/p} \le C\|f\|_{H^p(\mathbb{R}^n)},$$

where C is a positive constant independent of f and F (see [86, p. 128]).

Recently, Sawano et al. [81] originally introduced the ball quasi-Banach function space X which further generalizes the Banach function space in [3] in order to include weighted Lebesgue spaces, Morrey spaces, mixed-norm Lebesgue spaces, Orlicz-slice spaces, and Musielak–Orlicz spaces. Observe that the aforementioned several function spaces are not quasi-Banach function spaces which were originally introduced in [3, Definitions 1.1 and 1.3]; see, for instance, [81, 84, 96, 104]. In the same article [81], Sawano et al. also introduced the Hardy space $H_X(\mathbb{R}^n)$, associated with X, and established its various maximal function characterizations by assuming that the Hardy–Littlewood maximal operator is bounded on the p-convexification of X, as well as its several other real-variable characterizations, respectively, in terms of atoms, molecules, and Lusin area functions by assuming that the Hardy–Littlewood maximal operator satisfies a Fefferman–Stein vector-valued inequality on X and is bounded on the associate space of X.

Later, Wang et al. [95] further established the Littlewood–Paley g-function and the Littlewood–Paley g_λ^*-function characterizations of both $H_X(\mathbb{R}^n)$ and its local version $h_X(\mathbb{R}^n)$ and obtained the boundedness of anisotropic Calderón–Zygmund operators and pseudo-differential operators, respectively, on $H_X(\mathbb{R}^n)$ and $h_X(\mathbb{R}^n)$; Yan et al. [100] established the dual theorem and the intrinsic square function characterizations of $H_X(\mathbb{R}^n)$; Zhang et al. [103] introduced some new ball Campanato-type function space which proves the dual space of $H_X(\mathbb{R}^n)$ and established its Carleson measure characterization. Very recently, on spaces $\mathcal{X}$ of homogeneous type, Yan et al. [99, 98] introduced ball quasi-Banach function spaces $Y(\mathcal{X})$ and Hardy-type spaces $H_Y(\mathcal{X})$, associated with $Y(\mathcal{X})$, and developed a complete real-variable theory of $H_Y(\mathcal{X})$. For more studies about ball quasi-Banach function spaces, this thesis refers the reader to [17, 51, 52, 80, 87, 90, 101].

On the other hand, starting from 1970's, there has been an increasing interesting in extending classical function spaces arising in harmonic analysis from $\mathbb{R}^n$ to various anisotropic settings and some other domains; see, for instance, [22, 36, 38, 39, 42, 77, 85, 89, 91, 92]. The study of function spaces on $\mathbb{R}^n$ associated with anisotropic dilations was originally started from the celebrated articles [13, 14, 15] of Calderón and Torchinsky on anisotropic Hardy spaces. In 2003, Bownik [4] introduced and investigated the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ with $p \in (0,\infty)$, where A is a general expansive matrix on $\mathbb{R}^n$. Since then, various variants of classical Hardy spaces over the anisotropic Euclidean space have been introduced and their real-variable theories have been systematically developed. To be precise, Bownik et al. [7] further extended the anisotropic Hardy space to the weighted setting. Li et al. [59] introduced the anisotropic Musielak–Orlicz Hardy space $H_A^\varphi(\mathbb{R}^n)$, where φ is an anisotropic Musielak–Orlicz function, and characterized $H_A^\varphi(\mathbb{R}^n)$ by several maximal functions and atoms. Liu et al. [74, 76] first introduced the anisotropic Hardy–Lorentz space $H^{p,q}_A(\mathbb{R}^n)$, with $p \in (0,1]$ and $q \in (0,\infty]$, and established their several real-variable characterizations, respectively, in terms of atoms or finite atoms, molecules, maximal functions, and Littlewood–Paley functions, which are further applied to obtain the real interpolation theorem of $H^{p,q}_A(\mathbb{R}^n)$ and the boundedness of anisotropic Calderón–Zygmund operators on $H^{p,q}_A(\mathbb{R}^n)$ including the critical case. Liu et al. [66, 72] and Huang et al. [47] further generalized the corresponding results in [74, 76] to variable Hardy spaces and mixed-norm Hardy spaces, respectively. Recently, Liu et al. [73, 75] introduced the anisotropic variable Hardy–Lorentz space $H^{p(\cdot),q}_A(\mathbb{R}^n)$, where $p(\cdot) : \mathbb{R}^n \to (0,\infty]$ is a variable exponent function satisfying the globally log-Hölder continuous condition and $q \in (0,\infty]$, and developed a complete real-variable theory of these spaces including various equivalent characterizations and the boundedness of sublinear operators. Independently, Almeida et al. [1] also investigated the anisotropic variable Hardy–Lorentz space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, where $p(\cdot)$ and $q(\cdot)$ are nonnegative measurable functions on $(0,\infty)$. In [1], equivalent characterizations of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ in terms of maximal functions and atoms were established. It is remarkable that the anisotropic variable Hardy–Lorentz space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ in [1] and that in [73, 75] can not cover each other because the variable exponent $p(\cdot)$ in [1] is only defined on $(0,\infty)$, instead of $\mathbb{R}^n$. Particularly, Huang et al. [48, 49] further enriched the real-variable theory of anisotropic mixed-norm Campanato spaces and anisotropic variable Campanato spaces and established the dual theory of both anisotropic Hardy spaces $H^{\vec{p}}_A(\mathbb{R}^n)$ and $H^{p(\cdot)}_A(\mathbb{R}^n)$ with the full ranges of both $\vec{p}$ and $p(\cdot)$. For more studies about function spaces on the anisotropic Euclidean space, this thesis refers the reader to [5, 8, 20, 21, 56, 57].

Recall that the anisotropic Hardy space $H^A_X(\mathbb{R}^n)$ associated with both A and X was first introduced and studied by Wang et al. [97], in which they characterized $H^A_X(\mathbb{R}^n)$ in terms of maximal functions, atoms, finite atoms, and molecules and obtained the boundedness of the anisotropic Calderón–Zygmund operators on $H^A_X(\mathbb{R}^n)$. Motivated by this and [103], a quite *natural question* arises: can this thesis proves whether or not the dual space of $H^A_X(\mathbb{R}^n)$ is the anisotropic ball Campanato function space and characterize this space by the Carleson measure? The main target of this article is to give an affirmative answer to this question. Indeed, to answer this question and also to enrich the real-variable theory of anisotropic ball Campanato spaces as well as anisotropic Hardy spaces associated with both A and X, in this article, by borrowing some ideas from [103], namely considering finite linear combinations of atoms as a whole instead of a single atom, this thesis introduces the anisotropic ball Campanato function space and give some applications. Using this and the additional assumptions that the Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued inequality on X and is bounded on the associate space of X, this thesis gets rid of the dependence on the concavity of $\|\cdot\|_X$ and prove that the dual space of $H^A_X(\mathbb{R}^n)$ is just the anisotropic ball Campanato function space. From this, this thesis further deduces several equivalent characterizations of anisotropic ball Campanato function spaces. Moreover, via embedding X into a certain anisotropic weighted Lebesgue space, this thesis overcomes the difficulty caused by the absence of both an explicit expression and the absolute continuity of the quasi-norm $\|\cdot\|_X$ under consideration and establish the anisotropic Littlewood–Paley characterizations of $H^A_X(\mathbb{R}^n)$, which, together with the dual theorem of $H^A_X(\mathbb{R}^n)$ and the atomic decomposition of anisotropic tent spaces associated with X, finally implies the Carleson measure characterization of anisotropic ball Campanato function spaces. Moreover, this thesis shows that the Fourier transform $\widehat{f}$ of $f \in H^A_X(\mathbb{R}^n)$ coincides with a continuous function F on $\mathbb{R}^n$ in the sense of tempered distributions and prove that an inequality similar to (1.1.1) also holds for any $f \in H^A_X(\mathbb{R}^n)$. Furthermore, applying this and a technical inequality about the value of the Fourier transform of atoms, this thesis further conclude a higher order convergence of the continuous function F at the origin and then show that an inequality similar to (1.1.2) holds for $H^A_X(\mathbb{R}^n)$, which is a variant of the Hardy–Littlewood inequality in $H^A_X(\mathbb{R}^n)$.

The remainder of this article is organized as follows.

In Section 1.2, this thesis recall some notation and concepts which are used throughout this article. More precisely, this thesis first recall the definitions of the expansive matrix A, the step homogeneous quasi-norm ρ, and the ball quasi-Banach function space X. Then this thesis make some mild assumptions on the boundedness of the Hardy–Littlewood maximal operator on both X and its associate space, which are needed throughout this article. Finally, this thesis recall the concept of the non-tangential (grand) maximal function.

The aim of Chapter 2 is to give the dual space of the anisotropic Hardy space $H^A_X(\mathbb{R}^n)$ (see Theorem 2.2.6 below). To this end, in Section 2.1, this thesis first introduce the anisotropic ball Campanato function space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ (see Definition 2.1.3 below) and give an equivalent quasi-norm characterization of $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ (see Proposition 2.1.5 below). Using these, in Section 2.2 both the known atomic and the known finite atomic characterizations of $H^A_X(\mathbb{R}^n)$, and the assumptions that the Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued inequality on X and is bounded on the associate space of X, this thesis proves that the dual space of $H^A_X(\mathbb{R}^n)$ is just $\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$. At the end of this section, this thesis also give its invariance of $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ on different indices q and d; see Corollary 2.2.7 below.

In Chapter 3, this thesis establishes various real-variable characterizations of the anisotropic Hardy space $H^A_X(\mathbb{R}^n)$. Section 3.1 is devoted to establishing the anisotropic Littlewood–Paley function characterization of $H^A_X(\mathbb{R}^n)$, including the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g^*_λ-function, respectively, in Theorems 3.1.4, 3.1.5, and 3.1.6 below. This thesis first prove Theorem 3.1.4. To this end, this thesis first show that the quasi-norms in X of the anisotropic Lusin area functions defined by different Schwartz functions are equivalent (see Theorem 3.1.7 below). Then, via borrowing some ideas from [76] and the anisotropic Calderón reproducing formula (see Lemma 3.1.2 below), this thesis complete the proof of Theorem 3.1.4. From this and an approach initiated by Ullrich [93] and further developed by Liang et al. [65] and Liu et al. [73], together with Fefferman–Stein vector-valued inequality on X, this thesis obtains the anisotropic Littlewood–Paley g-function and the anisotropic Littlewood–Paley g_λ^*-function characterizations. The aim of Section 3.2 is to prove the Fourier transform $\widehat{f}$ of $f \in H_X^A(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions (see Theorem 3.2.1). In order to achieve this, this thesis applies Lemmas 3.2.5 (some subtle estimates on derivatives of the Fourier transform of the dilation of atoms) and 3.2.7 (some exquisite relations between the Euclidean norm and the step homogeneous quasi-norm ρ under consideration) to establish a uniform pointwise estimate for atoms (see Lemma 3.2.6). Then Theorem 3.2.1 is proved by this and some real-variable characterizations from [97], especially the atomic decomposition of $H_X^A(\mathbb{R}^n)$. Applying the Fourier transform, in Section 3.3, this thesis presents some further applications of Theorem 3.2.1. First, this thesis proves that the above function F has a higher order convergence at the origin (see Theorem 3.3.1). Second, this thesis shows that

$$|F(\cdot)| \min\left\{[\rho_*(\cdot)]^{1-\frac{1}{p_-}-\frac{1}{q_0}+(d+1)\frac{\ln\lambda_-}{\ln b}},\ [\rho_*(\cdot)]^{1-\frac{2}{q_0}+(d+1)\frac{\ln\lambda_-}{\ln b}}\right\}$$

is $L^{q_0}(\mathbb{R}^n)$-integrable and a positive constant multiple of the anisotropic Hardy space norm of f can uniformly control this integral. Thus, this thesis extends the Hardy–Littlewood inequality to the setting of anisotropic Hardy spaces associated with ball quasi-Banach function spaces (see Theorem 3.3.2).

In Chapter 4, this thesis establishes various real-variable characterizations of the anisotropic ball Campanato space $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$. In Section 4.1, by applying the dual result obtained in Theorem 2.2.6 and a key estimate (see Lemma 4.1.2 below), this thesis obtains several equivalent characterizations of $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$ (see Theorems 4.1.1 and 4.1.3 below), which are further applied to establish the Carleson measure characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ in the next section. In Section 4.2, this thesis establishes the Carleson measure characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ (see Theorem 4.2.3 below). Indeed, via using Theorems 2.2.6, 4.1.1, and 3.1.4, as well as the atomic decomposition of anisotropic tent spaces associated with X (see Lemma 4.2.7 below), this thesis shows that a measurable function h belongs to $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ if and only if h generates an X–Carleson measure $d\mu$. Moreover, the norm of the X–Carleson measure $d\mu$ is equivalent to the $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$-norm of h.

In Section 5, this thesis applies all the main results obtained in the above sections to several specific ball quasi-Banach function spaces, via verifying all necessary assumptions, to seven concrete examples of ball quasi-Banach function spaces, respectively, Morrey spaces (see section 5.1 below), Orlicz-slice spaces (see Subsection 5.2 below), Lorentz spaces (see section 5.3 below), variable Lebesgue spaces (see section 5.4 below), mixed-norm Lebesgue spaces (see Subsection 5.5 below), weighted Lebesgue spaces (see section 5.6 below), and Orlicz spaces (see section 5.7 below). Moreover, in Subsection 5.1, this thesis give an example to show the limitation of Theorems 3.2.1, 3.3.1 and 3.3.2 on the

Morrey space because its norm lacks concavity.

1.2 Notation

c1s2

In this section, this thesis first makes some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, $\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, and $\mathbf{0}$ be the *origin* of $\mathbb{R}^n$. For any multi-index $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and any $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$, and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. This thesis denotes by C a *positive constant* which is independent of the main parameters involved, but may vary from line to line. This thesis use $C_{(\alpha,\ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \ldots$. The symbol $f \lesssim g$ means $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, this thesis then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, this thesis then write $f \lesssim g = h$ or $f \lesssim g \le h$. For any $q \in [1, \infty]$, this thesis denotes by q' its *conjugate index*, that is, $1/q + 1/q' = 1$. For any $x \in \mathbb{R}^n$, this thesis denotes by $|x|$ the n-dimensional *Euclidean metric* of x. If E is a subset of $\mathbb{R}^n$, this thesis denotes by $\mathbf{1}_E$ its *characteristic function* and by $E^\complement$ the set $\mathbb{R}^n \setminus E$. For any $r \in (0, \infty)$ and $x \in \mathbb{R}^n$, this thesis denotes by $B(x, r)$ the ball centered at x with the radius r, that is, $B(x, r) := \{y \in \mathbb{R}^n : \ |x - y| < r\}$. For any ball B, this thesis use x_B to denote its center and r_B its radius and this thesis denotes by λB for any $\lambda \in (0, \infty)$ the ball concentric with B having the radius λr_B. This thesis also use $\epsilon \to 0^+$ to denote that there exists $\alpha_0 \in (0, \infty)$ such that $\epsilon \in (0, \alpha_0)$ and $\epsilon \to 0$. Let X and Y be two normed vector spaces, respectively, with the norm $\|\cdot\|_X$ and the norm $\|\cdot\|_Y$; then this thesis use $X \hookrightarrow Y$ to denote $X \subset Y$ and there exists a positive constant C such that, for any $f \in X$, $\|f\|_Y \le C\|f\|_X$. At last, when this thesis proves a theorem (or the like), this thesis always use the same symbols as those used in the statement itself of that theorem (or the like).

Then this thesis recall some notation and concepts on dilations (see, for instance [4, 43]) as well as ball quasi-Banach function spaces (see, for instance, [60, 61, 81, 95, 96, 100, 104]). This thesis begins with recalling the concept of the expansive matrix from [4].

dilation **Definition 1.2.1.** A real $n \times n$ matrix A is called an *expansive matrix* (shortly, a *dilation*) if

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

here and thereafter, $\sigma(A)$ denotes the *set of all eigenvalues of* A.

Let A be a dilation and

2.14.x1 (1.2.1)
$$b := |\det A|,$$

where $\det A$ denotes the determinant of A. Let $A := (a_{i,j})_{1 \le i,j \le n}$ be a dilation, then the matrix norm is defined as

$$||A|| := (\sum_{i,j=1}^{n} |a_{i,j}|^2)^{1/2}.$$

Then it follows from [4, p. 6, (2.7)] that $b \in (1, \infty)$. By the fact that there exists an open and symmetry ellipsoid Δ, with $|\Delta| = 1$, and an $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$ (see [4, p. 5, Lemma 2.2]), this thesis finds that, for any $k \in \mathbb{Z}$,

B_k (1.2.2)
$$B_k := A^k \Delta$$

is open, $B_k \subset rB_k \subset B_{k+1}$, and $|B_k| = b^k$. For any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, the ellipsoid $x + B_k$ is called a *dilated ball*. In what follows, this thesis always lets $\mathcal{B}$ be the set of all such dilated balls, that is,

ball-B (1.2.3)
$$\mathcal{B} := \{x + B_k : \ x \in \mathbb{R}^n, k \in \mathbb{Z}\}$$

and let

tau (1.2.4)
$$\tau := \inf \left\{ l \in \mathbb{Z} : \ r^l \geq 2 \right\}.$$

Let $\lambda_-, \lambda_+ \in (0, \infty)$ satisfy that

2.21.x1 (1.2.5)
$$1 < \lambda_- < \min\{|\lambda| : \ \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \ \lambda \in \sigma(A)\} < \lambda_+.$$

This thesis point out that, if A is diagonalizable over $\mathbb{R}$, then this thesis may let

$$\lambda_- := \min\{|\lambda| : \ \lambda \in \sigma(A)\} \text{ and } \lambda_+ := \max\{|\lambda| : \ \lambda \in \sigma(A)\}.$$

Otherwise, this thesis may choose them sufficiently close to these equalities in accordance with what this thesis needs in the arguments.

The following definition of the homogeneous quasi-norm is just [4, p. 6, Definition 2.3].

quasi-norm **Definition 1.2.2.** A *homogeneous quasi-norm*, associated with a dilation A, is a measurable mapping $\varrho : \ \mathbb{R}^n \to [0, \infty)$ such that

(i) $\varrho(x) = 0 \Longleftrightarrow x = \mathbf{0}$, where $\mathbf{0}$ denotes the origin of $\mathbb{R}^n$;

(ii) $\varrho(Ax) = b\varrho(x)$ for any $x \in \mathbb{R}^n$;

(iii) there exists an $A_0 \in [1, \infty)$ such that, for any $x, y \in \mathbb{R}^n$,

$$\varrho(x + y) \leq A_0 \left[\varrho(x) + \varrho(y)\right].$$

In the standard Euclidean space case, let $A := 2\, I_{n\times n}$ and, for any $x \in \mathbb{R}^n$, $\varrho(x) := |x|^n$. Then ϱ is an example of homogeneous quasi-norms associated with A on $\mathbb{R}^n$. Here and thereafter, $I_{n\times n}$ always denotes the $n \times n$ *unit matrix* and $|\cdot|$ the *Euclidean norm* in $\mathbb{R}^n$.

For a fixed dilation A, by [4, p. 6, Lemma 2.4], this thesis defines the following quasi-norm which is used throughout this article.

def-shqn **Definition 1.2.3.** Define the *step homogeneous quasi-norm* ρ on $\mathbb{R}^n$, associated with the dilation A, by setting

$$\rho(x) := \begin{cases} b^k & \text{if } x \in B_{k+1} \backslash B_k, \\ 0 & \text{if } x = \mathbf{0}, \end{cases}$$

where b is the same as in (1.2.1) and, for any $k \in \mathbb{Z}$, B_k the same as in (1.2.2).

Then $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [25], where dx denotes the n-dimensional Lebesgue measure. For more studies on the real-variable theory of function spaces over spaces of homogeneous type, this thesis refers the reader to [10, 11, 12, 62, 63, 64].

Throughout this article, this thesis always lets A be a dilation in Definition 1.2.1, b the same as in (1.2.1), ρ the step homogeneous quasi-norm in Definition 1.2.3, $\mathcal{B}$ the set of all dilated balls in (1.2.3), $\mathscr{M}(\mathbb{R}^n)$ the *set of all measurable functions* on $\mathbb{R}^n$, and B_k for any $k \in \mathbb{Z}$ the same as in (1.2.2). Now, this thesis recall the definition of ball quasi-norm Banach function spaces (see [81]).

BQBFS **Definition 1.2.4.** A quasi-normed linear space $X \subset \mathscr{M}(\mathbb{R}^n)$, equipped with a quasi-norm $\|\cdot\|$ which makes sense for the whole $\mathscr{M}(\mathbb{R}^n)$, is called a *ball quasi-Banach function space* if it satisfies

(i) for any $f \in \mathscr{M}(\mathbb{R}^n)$, $\|f\|_X = 0$ implies that $f = 0$ almost everywhere;

(ii) for any $f, g \in \mathscr{M}(\mathbb{R}^n)$, $|g| \le |f|$ almost everywhere implies that $\|g\|_X \le \|f\|_X$;

(iii) for any $\{f_m\}_{m\in\mathbb{N}} \subset \mathscr{M}(\mathbb{R}^n)$ and $f \in \mathscr{M}(\mathbb{R}^n)$, $0 \le f_m \uparrow f$ as $m \to \infty$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$ as $m \to \infty$;

(iv) $\mathbf{1}_B \in X$ for any dilated ball $B \in \mathcal{B}$.

Moreover, a ball quasi-Banach function space X is called a *ball Banach function space* if it satisfies:

(v) for any $f, g \in X$, $\|f + g\|_X \le \|f\|_X + \|g\|_X$;

(vi) for any given dilated ball $B \in \mathcal{B}$, there exists a positive constant $C_{(B)}$ such that, for any $f \in X$,

$$\int_B |f(x)|\, dx \le C_{(B)} \|f\|_X.$$

s1r1 **Remark 1.2.5.** (i) As was mentioned in [97, Remark 2.5(i)], if $f \in \mathscr{M}(\mathbb{R}^n)$, then $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere; if $f, g \in \mathscr{M}(\mathbb{R}^n)$ and $f = g$ almost everywhere, then $\|f\|_X \sim \|g\|_X$ with the positive equivalence constants independent of both f and g.

(ii) As was mentioned in [97, Remark 2.5(ii)], if this thesis replaces any dilated ball $B \in \mathcal{B}$ in Definition 1.2.4 by any bounded measurable set E or by any ball $B(x, r)$ with $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, this thesis obtains its another equivalent formulation.

(iii) By [30, Theorem 2], this thesis finds that both (ii) and (iii) of Definition 1.2.4 imply that any ball quasi-Banach function space is complete.

Now, this thesis recall the concepts of the p-convexification and the concavity of ball quasi-Banach function spaces, which is a part of [81, Definition 2.6].

Debf **Definition 1.2.6.** Let X be a ball quasi-Banach function space and $p \in (0, \infty)$.

(i) The *p-convexification* X^p of X is defined by setting

$$X^p := \{f \in \mathscr{M}(\mathbb{R}^n) : \ |f|^p \in X\}$$

equipped with the *quasi-norm* $\|f\|_{X^p} := \||f|^p\|_X^{1/p}$.

(ii) The space X is said to be *concave* if there exists a positive constant C such that, for any $\{f_k\}_{k\in\mathbb{N}} \subset \mathscr{M}(\mathbb{R}^n)$,

$$\sum_{k=1}^{\infty} \|f_k\|_X \leq C \left\| \sum_{k=1}^{\infty} |f_k| \right\|_X .$$

In particular, when $C = 1$, X is said to be *strictly concave*.

The associate space X' of any given ball Banach function space X is defined as follows; see [3, Chapter 1, Section 2] or [81, p. 9].

Definition 1.2.7. For any given ball Banach function space X, its *associate space* (also called the *Köthe dual space*) X' is defined by setting

$$X' := \left\{ f \in \mathscr{M}(\mathbb{R}^n) : \ \|f\|_{X'} := \sup_{g\in X, \ \|g\|_X = 1} \|fg\|_{L^1(\mathbb{R}^n)} < \infty \right\},$$

where $\|\cdot\|_{X'}$ is called the *associate norm* of $\|\cdot\|_X$.

bbf **Remark 1.2.8.** From [81, Proposition 2.3], this thesis deduces that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space.

Next, this thesis recall the concept of absolutely continuous quasi-norms of X as follows (see [95, Definition 3.2] for the standard Euclidean space case and [99, Definition 6.1] for the case of spaces of homogeneous type).

Definition 1.2.9. Let X be a ball quasi-Banach function space. A function $f \in X$ is said to have an *absolutely continuous quasi-norm* in X if $\|f\mathbf{1}_{E_j}\|_X \downarrow 0$ whenever $\{E_j\}_{j=1}^\infty$ is a sequence of measurable sets satisfying $E_j \supset E_{j+1}$ for any $j \in \mathbb{N}$ and $\bigcap_{j=1}^\infty E_j = \emptyset$. Moreover, X is said to have an *absolutely continuous quasi-norm* if, for any $f \in X$, f has an absolutely continuous quasi-norm in X.

Now, this thesis recall the concept of the Hardy–Littlewood maximal operator. Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ denote the *set of all locally integrable functions* on $\mathbb{R}^n$. Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}(f)$ of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} \fint_{y+B_k} |f(z)|\,dz = \sup_{x\in B\in\mathcal{B}} \fint_B |f(z)|\,dz,$$

where $\mathcal{B}$ is the same as in (1.2.3) and the last supremum is taken over all balls $B \in \mathcal{B}$. For any given $\alpha \in (0, \infty)$, the *powered Hardy–Littlewood maximal operator* $\mathcal{M}^{(\alpha)}$ is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\alpha)}(f)(x) := \{\mathcal{M}\left(|f|^\alpha\right)(x)\}^{\frac{1}{\alpha}}.$$

Throughout this article, this thesis also need the following two fundamental assumptions about the boundedness of $\mathcal{M}$ on the given ball quasi-Banach function space and its associate space.

Assum-1 **Assumption 1.2.10.** Let X be a ball quasi-Banach function space. Assume that there exists a $p_- \in (0, \infty)$ such that, for any $p \in (0, p_-)$ and $u \in (1, \infty)$, there exists a positive constant C, depending on both p and u, such that, for any $\{f_k\}_{k=1}^\infty \subset \mathscr{M}(\mathbb{R}^n)$,

$$\left\|\left\{\sum_{k=1}^\infty \left[\mathcal{M}\left(f_k\right)\right]^u\right\}^{\frac{1}{u}}\right\|_{X^{\frac{1}{p}}} \le C \left\|\left\{\sum_{k=1}^\infty |f_k|^u\right\}^{\frac{1}{u}}\right\|_{X^{\frac{1}{p}}}.$$

Remark 1.2.11. Let X be a ball-Banach function space. Suppose that $\mathcal{M}$ is bounded on X and that $\mathcal{M}$ is bounded on X'. By an argument similar to that used in the proof of [26, Theorem 4.10], this thesis obtains that $\mathcal{M}$ satisfies Assumption 1.2.10 with $p_- = 1$.

In what follows, for any given $p_- \in (0, \infty)$, this thesis always lets

underp (1.2.6)
$$\underline{p} := \min\{p_-, 1\}.$$

Assum-2 **Assumption 1.2.12.** Let $p_- \in (0, \infty)$ and X be a ball quasi-Banach function space. Assume that there exists a $\theta_0 \in (0, \underline{p})$, with $\underline{p}$ the same as in (1.2.6), and a $p_0 \in (\theta_0, \infty)$ such that X^{1/θ_0} is a ball Banach function space and, for any $f \in (X^{1/\theta_0})'$,

$$\left\|\mathcal{M}^{((p_0/\theta_0)')}(f)\right\|_{(X^{1/\theta_0})'} \le C\|f\|_{(X^{1/\theta_0})'},$$

where C is a positive constant, independent of f, and $\frac{1}{p_0/\theta_0} + \frac{1}{(p_0/\theta_0)'} = 1$.

Next, recall that a *Schwartz function* is a function $\varphi \in C^\infty(\mathbb{R}^n)$ satisfying that, for any $k \in \mathbb{Z}_+$ and any multi-index $\alpha \in \mathbb{Z}^n_+$,

6.11.x1 (1.2.7)
$$\|\varphi\|_{\alpha,k} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^k |\partial^\alpha \varphi(x)| < \infty.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *set of all Schwartz functions*, equipped with the topology determined by $\{\|\cdot\|_{\alpha,k}\}_{\alpha\in\mathbb{Z}_+^n,k\in\mathbb{Z}_+}$. Then $\mathcal{S}'(\mathbb{R}^n)$ is defined to be the *dual space* of $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak-$*$ topology. For any $N\in\mathbb{Z}_+$, let

$$\mathcal{S}_N(\mathbb{R}^n) := \{\varphi\in\mathcal{S}(\mathbb{R}^n):\ \|\varphi\|_{\alpha,k}\le 1, |\alpha|\le N, k\le N\},$$

equivalently,

$$\varphi\in\mathcal{S}_N(\mathbb{R}^n)$$
$$\iff \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}\max\{1,[\rho(x)]^N\}|\partial^\alpha\varphi(x)|\le 1.$$

In what follows, for any $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $k\in\mathbb{Z}$, let $\varphi_k(\cdot):=b^{-k}\varphi(A^{-k}\cdot)$.

Definition 1.2.13. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $f\in\mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function* $M_\varphi(f)$ with respect to φ is defined by setting, for any $x\in\mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{k\in\mathbb{Z},\, y\in x+B_k}|f*\varphi_k(y)|.$$

Moreover, for any given $N\in\mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ is defined by setting, for any $x\in\mathbb{R}^n$,

M_N

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x). \tag{1.2.8}$$

Chapter 2

Duality between $H_X^A(\mathbb{R}^n)$ and $\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$

c2

2.1 Definition of Anisotropic Ball Campanato Function Spaces

c2s1

In this section, this thesis provides a description of the dual space of the anisotropic Hardy space $H_X^A(\mathbb{R}^n)$ associated with ball quasi-Banach function space X. This description is a consequence of the definition of the anisotropic ball Campanato function space, the atomic and the finite atomic characterizations of $H_X^A(\mathbb{R}^n)$ from [97], as well as some basic tools from functional analysis. To state the dual theorem, this thesis first present the definition of $H_X^A(\mathbb{R}^n)$ from [97] as follows. In what follows, for any $\alpha \in \mathbb{R}$, this thesis denotes by $\lfloor \alpha \rfloor$ the largest integer not greater than α.

HXA **Definition 2.1.1.** Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 1.2.10 with $p_- \in (0,\infty)$ and Assumption 1.2.12 with the same p_-, $\theta_0 \in (0,\underline{p})$, and $p_0 \in (\theta_0,\infty)$, where $\underline{p}$ is the same as in (1.2.6). Assume that

3.14.x1 (2.1.1)
$$N \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2, \infty\right).$$

The *anisotropic Hardy space* $H_{X,N}^A(\mathbb{R}^n)$, associated with both A and X, is defined by setting

$$H_{X,N}^A(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) :\ \|M_N(f)\|_X < \infty\right\},$$

where $M_N(f)$ is the same as in (1.2.8). Moreover, for any $f \in H_{X,N}^A(\mathbb{R}^n)$, let

$$\|f\|_{H_{X,N}^A(\mathbb{R}^n)} := \|M_N(f)\|_X.$$

Let A be a dilation and X the same as in Definition 2.1.1. In the remainder of this article, this thesis always lets

NXA (2.1.2)
$$N_{X,A} := \left\lfloor \left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2.$$

s2r1 **Remark 2.1.2.** (i) As was mentioned in [97, Remark 2.17(i)], the space $H^A_{X,N}(\mathbb{R}^n)$ is independent of the choice of N as long as $N\in\mathbb{N}\cap[N_{X,A},\infty)$. In what follows, when $N\in\mathbb{N}\cap[N_{X,A},\infty)$, this thesis writes $H^A_{X,N}(\mathbb{R}^n)$ simply by $H^A_X(\mathbb{R}^n)$.

(ii) This thesis point out that, if $A:=2\,I_{n\times n}$, then $H^A_X(\mathbb{R}^n)$ coincides with $H_X(\mathbb{R}^n)$ which was introduced by Sawano et al. in [81].

In what follows, for any $d\in\mathbb{Z}_+$, $\mathcal{P}_d(\mathbb{R}^n)$ denotes the set of all the polynomials on $\mathbb{R}^n$ with degree not greater than d; for any ball $B\in\mathcal{B}$ and any locally integrable function g on $\mathbb{R}^n$, this thesis use $P^d_B g$ to denote the *minimizing polynomial* of g with degree not greater than d, which means that $P^d_B g$ is the unique polynomial $f\in\mathcal{P}_d(\mathbb{R}^n)$ such that, for any $h\in\mathcal{P}_d(\mathbb{R}^n)$,

$$\int_B [g(x)-f(x)]h(x)\,dx=0.$$

Next, this thesis introduces the anisotropic ball Campanato function space associated with the ball quasi-Banach function space. In what follows, this thesis use $L^q_{\rm loc}(\mathbb{R}^n)$ to denote the *set of all q-order locally integrable functions* on $\mathbb{R}^n$.

LAXqds **Definition 2.1.3.** Let A be a dilation, X a ball quasi-Banach function space, $q\in[1,\infty)$, $d\in\mathbb{Z}_+$, and $s\in(0,\infty)$. Then the *anisotropic ball Campanato function space* $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$, associated with X, is defined to be the set of all the $f\in L^q_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}:=\sup\left\|\left\{\sum_{i=1}^m\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac1s}\right\|_X^{-1}$$
$$\times\sum_{j=1}^m\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}\left|f(x)-P^d_{B^{(j)}}f(x)\right|^q\,dx\right]^{\frac1q}$$

is finite, where the supremum is taken over all $m\in\mathbb{N}$, $\{B^{(j)}\}_{j=1}^m\subset\mathcal{B}$, and $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$.

s2r2 **Remark 2.1.4.** Let A, X, q, d, and s be the same as in Definition 2.1.3.

(i) If this thesis has the basic assumption that $\|\{\sum_{i=1}^m[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}]^s\mathbf{1}_{B^{(i)}}\}^{\frac1s}\|_X^{-1}\in(0,\infty)$, the indice m in Definition 2.1.3 can be chosen as ∞; see Proposition 2.1.5 below.

(ii) Obviously, $\mathcal{P}_d(\mathbb{R}^n)\subset\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$. Indeed, $\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}=0$ if and only if $f\in\mathcal{P}_d(\mathbb{R}^n)$. Throughout this article, this thesis always identify $f\in\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ with $\{f+P:\ P\in\mathcal{P}_d(\mathbb{R}^n)\}$.

(iii) For any $f\in L^q_{\rm loc}(\mathbb{R}^n)$, define

$$|||f|||_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}:=\sup\inf\left\|\left\{\sum_{i=1}^m\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac1s}\right\|_X^{-1}$$

$$\times \sum_{j=1}^{m} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} |f(x) - P(x)|^q \, dx \right]^{\frac{1}{q}},$$

where the supremum is taken the same way as in Definition 2.1.3 and the infimum is taken over all $P \in \mathcal{P}_d(\mathbb{R}^n)$. Then, similarly to the proof of [100, Lemma 2.6] with using [4, p. 49, (8.9)] instead of [100, Lemma 2.5], this thesis easily find that $|||\cdot|||_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}$ is an equivalent quasi-norm of $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$.

Moreover, for the anisotropic ball Campanato function space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$, this thesis has the following equivalent quasi-norm.

s2p1 **Proposition 2.1.5.** *Let A, X, q, d, and s be the same as in Definition 2.1.3. For any $f \in L^q_{\rm loc}(\mathbb{R}^n)$, define*

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} := \sup \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \times \sum_{j\in\mathbb{N}} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}},$$

where the supremum is taken over all $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ satisfying that

s2e1 (2.1.3)
$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^s \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{s}} \right\|_X \in (0,\infty).$$

Then, for any $f \in L^q_{\rm loc}(\mathbb{R}^n)$,

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}.$$

Proof. Let $f \in L^q_{\rm loc}(\mathbb{R}^n)$. Obviously,

2.15.x1 (2.1.4)
$$\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} \le \widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}.$$

Conversely, let $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ satisfy (2.1.3). From Definition 1.2.4(iii), it follows that

$$\lim_{m\to\infty} \left\| \left\{ \sum_{i=1}^{m} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \times \sum_{j=1}^{m} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \\
&\quad \times \sum_{j\in\mathbb{N}} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}}.
\end{aligned}
$$

Therefore, for any $\varepsilon \in (0,\infty)$, there exists an $m_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
&\left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \\
&\quad \times \sum_{j\in\mathbb{N}} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}} \\
&< \left\| \left\{ \sum_{i=1}^{m_0} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \\
&\quad \times \sum_{i=1}^{m_0} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}} + \varepsilon \\
&\le \|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} + \varepsilon,
\end{aligned}
$$

which, together with the arbitrariness of both $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ satisfying (2.1.3) and $\varepsilon \in (0,\infty)$, further implies that

$$
\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} \le \|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}.
$$

This, combined with (2.1.4), then finishes the proof of Proposition 2.1.5. □

Then this thesis introduces another anisotropic ball Campanato function space $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ associated with the ball quasi-Banach function space X.

cqdA **Definition 2.1.6.** Let A be a dilation, X a ball quasi-Banach function space, $q \in [1,\infty)$, and $d \in \mathbb{Z}_+$. Then the *Campanato space* $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$, associated with both A and X, is defined to be the set of all the $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$
\|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \frac{|B|}{\|\mathbf{1}_B\|_X} \left\{ \fint_B \left| f(x) - P^d_B f(x) \right|^q \, dx \right\}^{\frac{1}{q}} < \infty,
$$

where the supremum is taken over all balls $B \in \mathcal{B}$ and $P^d_B f$ denotes the minimizing polynomial of f with degree not greater than d.

s2r3 **Remark 2.1.7.** Let A, X, q, d, and s be the same as in Definition 2.1.3.

(i) From Definitions 2.1.3 and 2.1.6, it immediately follows that $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n) \subset \mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ and this inclusion is continuous.

(ii) For any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$|||f|||_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \inf_{P\in\mathcal{P}_d(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_X} \left[\fint_B |f(x) - P(x)|^q \, dx \right]^{\frac{1}{q}}.$$

Then, similarly to [100, Lemma 2.6], this thesis finds that $|||\cdot|||_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)}$ is an equivalent quasi-norm of $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$.

Now, this thesis give a basic inequality which is used throughout this article.

basicine **Lemma 2.1.8.** *Let $\{a_i\}_{i\in\mathbb{N}} \subset [0,\infty)$. If $\alpha \in [1,\infty)$, then*

$$\left(\sum_{i\in\mathbb{N}} a_i\right)^\alpha \geq \sum_{i\in\mathbb{N}} a_i^\alpha.$$

The following proposition shows that, if the ball quasi-Banach function space X is concave and $s \in (0,1]$, then the space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ coincides with $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ introduced in Definition 2.1.6.

s2p2 **Proposition 2.1.9.** *Let X be a concave ball quasi-Banach function space, $q \in [1,\infty)$, $d \in \mathbb{Z}_+$, and $s \in (0,1]$. Then*

2.15.x3 (2.1.5)
$$\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n) = \mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Proof. This thesis first show that

2.15.x2 (2.1.6)
$$\mathcal{L}^A_{X,q,d}(\mathbb{R}^n) \subset \mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$$

and the inclusion is continuous. For this purpose, let $f \in \mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$. Then, from the assumptions that X is concave, Definitions 1.2.4(ii) and 1.2.6(ii), and $s \in (0,1]$, together with Lemma 2.1.8, this thesis deduces that

$$\begin{aligned}
\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} &\lesssim \sup \left(\sum_{i=1}^m \lambda_i\right)^{-1} \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \\
&\quad \times \left[\fint_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q dx \right]^{\frac{1}{q}} \\
&\leq \sup \left(\sum_{i=1}^m \lambda_i\right)^{-1} \sum_{j=1}^m \lambda_j \|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)},
\end{aligned}$$

where the supremum is taken over all $m \in \mathbb{N}$, $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$, and $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$. This further implies (2.1.6). Combining (2.1.6) and Remark 2.1.7(i), this thesis obtains (2.1.5), which completes the proof of Proposition 2.1.9. □

2.2 Duatlity Theorem

c2s2

In this section, this thesis establishes the relation between $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ and $H^A_X(\mathbb{R}^n)$. To this end, this thesis first recall the definitions of the anisotropic (X,q,d)-atom and the anisotropic finite atomic Hardy space $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ from [97, Definitions 4.1 and 5.1].

deffin **Definition 2.2.1.** Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 1.2.10 with $p_- \in (0,\infty)$ and Assumption 1.2.12 with the same p_-, $\theta_0 \in (0,\underline{p})$, and $p_0 \in (\theta_0,\infty)$, where $\underline{p}$ is the same as in (1.2.6). Assume that $N \in \mathbb{N}\cap[N_{X,\,A},\infty)$ with $N_{X,\,A}$ the same as in (2.1.2). Further assume that $q \in (\max\{p_0,1\},\infty]$ and

def-d (2.2.1)
$$d \in \left[\left\lfloor \left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor,\infty\right)\cap\mathbb{Z}_+.$$

(i) An *anisotropic* (X,q,d)*-atom* a is a measurable function on $\mathbb{R}^n$ satisfying that

(i)$_1$ $\operatorname{supp} a := \{x\in\mathbb{R}^n:\ a(x)\neq 0\}\subset B$, where $B\in\mathcal{B}$ and $\mathcal{B}$ is the same as in (1.2.3);

(i)$_2$ $\|a\|_{L^q(\mathbb{R}^n)} \le |B|^{\frac{1}{q}}\|\mathbf{1}_B\|_X^{-1}$;

(i)$_3$ $\int_{\mathbb{R}^n} a(x)x^\gamma\,dx = 0$ for any $\gamma\in\mathbb{Z}^n_+$ with $|\gamma|\le d$, here and thereafter, for any $\gamma := \{\gamma_1,\dots,\gamma_n\}\in\mathbb{Z}^n_+$, $|\gamma| := \gamma_1+\cdots+\gamma_n$ and $x^\gamma := x_1^{\gamma_1}\cdots x_n^{\gamma_n}$.

(ii) The *anisotropic finite atomic Hardy space* $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, associated with both A and X, is defined to be the set of all the $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying that there exists a $K\in\mathbb{N}$, a sequence $\{\lambda_i\}_{i=1}^K\subset(0,\infty)$, and a finite sequence $\{a_i\}_{i=1}^K$ of anisotropic (X,q,d)-atoms supported, respectively, in $\{B^{(i)}\}_{i=1}^K\subset\mathcal{B}$ such that

$$f=\sum_{i=1}^K \lambda_i a_i.$$

Moreover, for any $f\in H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)} := \inf\left\|\left\{\sum_{i=1}^K\left[\frac{\lambda_i\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X,$$

where the infimum is taken over all the decompositions of f as above.

Let A be a dilation and X the same as in Definition 2.2.1. In the remainder of this article, this thesis always lets

dxa (2.2.2)
$$d_{X,\,A} := \left\lfloor \left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor.$$

To establish the dual theorem of $H^A_X(\mathbb{R}^n)$, this thesis needs its atomic and its finite atomic characterizations as follows, which are simple corollaries of [97, Theorem 4.2 and Lemma 7.2] and [97, Theorem 5.4], respectively.

s2l1 **Lemma 2.2.2.** *Let A, X, q, d, and θ_0 be the same as in Definition 2.2.1. Further assume that X has an absolutely continuous quasi-norm, $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of anisotropic (X,q,d)-atoms supported, respectively, in the balls $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ such that*

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right]^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X<\infty.$$

Then the series $f:=\sum_{j\in\mathbb{N}}\lambda_j a_j$ converges in $H_X^A(\mathbb{R}^n)$, $f\in H_X^A(\mathbb{R}^n)$, and there exists a positive constant C, independent of f, such that

$$\|f\|_{H_X^A(\mathbb{R}^n)}\le C\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right]^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X.$$

finatomth **Lemma 2.2.3.** *Let A, X, q, d, θ_0, and p_0 be the same as in Definition 2.2.1.*

(i) *If $q\in(\max\{p_0,1\},\infty)$, then $\|\cdot\|_{H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_X^A(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)$;*

(ii) *$\|\cdot\|_{H_{X,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)}$ and $\|\cdot\|_{H_X^A(\mathbb{R}^n)}$ are equivalent quasi-norms on $H_{X,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$, where $\mathcal{C}(\mathbb{R}^n)$ denotes the set of all continuous functions on $\mathbb{R}^n$.*

The following conclusion is also needed for establishing the dual theorem.

atomch2 **Proposition 2.2.4.** *Let A, X, and d be the same as in Definition 2.2.1. Then the set $H_{X,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ is dense in $H_X^A(\mathbb{R}^n)$.*

Proof. From [97, Lemma 7.2], it easily follows that $H_{X,\,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)$ is dense in $H_X^A(\mathbb{R}^n)$. Thus, to show that $H_{X,\,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ is also dense in $H_X^A(\mathbb{R}^n)$, it suffices to prove that the set $H_{X,\,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ is dense in $H_{X,\,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)$ with the quasi-norm $\|\cdot\|_{H_X^A(\mathbb{R}^n)}$. To this end, this thesis only need to show that, for any given anisotropic (X,∞,d)-atom a supported in the anisotropic ball $B:=x_0+B_{i_0}$ with $x_0\in\mathbb{R}^n$ and $i_0\in\mathbb{Z}$,

phi_ka (2.2.3)
$$\lim_{k\to-\infty}\|a-\varphi_k*a\|_{H_X^A(\mathbb{R}^n)}=0,$$

where $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$ and $\operatorname{supp}\varphi\subset B_0$. Let $s\in(\max\{1,p_0\},\infty)$ with p_0 the same as in Definition 2.1.1. Observe that, for any $k\in(-\infty,0]\cap\mathbb{Z}$,

$$\frac{|B_{\max\{i_0,0\}+\tau}|^{\frac{1}{s}}(a-\varphi_k*a)}{\|\mathbf{1}_{x_0+B_{\max\{i_0,0\}+\tau}}\|_X\|a-\varphi_k*a\|_{L^s(\mathbb{R}^n)}}$$

is an anisotropic (X,s,d)-atom supported in the anisotropic ball $x_0+B_{\max\{i_0,0\}+\tau}$, which, combined with Lemma 2.2.2, further implies that

$$\begin{aligned}\|a-\varphi_k*a\|_{H_X^A(\mathbb{R}^n)} &\lesssim \frac{\|\mathbf{1}_{x_0+B_{\max\{i_0,0\}+\tau}}\|_X\|a-\varphi_k*a\|_{L^s(\mathbb{R}^n)}}{|B_{\max\{i_0,0\}+\tau}|^{\frac{1}{s}}}\\ &\lesssim \|a-\varphi_k*a\|_{L^s(\mathbb{R}^n)}.\end{aligned}$$

From this and [4, p.15, Lemma 3.8], this thesis deduces (2.2.3), which then completes the proof of Proposition 2.2.4. □

The following technical lemma is just [4, p. 49, (8.9)] (see also [70, Lemma 3.4]). For the sake of completeness, this thesis give its proof below.

s3l1 **Lemma 2.2.5.** *Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$, $d\in\mathbb{Z}_+$, and B be an anisotropic ball in $\mathcal{B}$. Then there exists a positive constant C, depending only on d, such that*

$$\sup_{x\in B}\left|P_B^d f(x)\right|\le C\fint_B |f(x)|\,dx.$$

Proof. Let $\{P_k\}_{k\in\{0,\cdots,d\}}$ be the standard orthogonal polynomials of $L^2(B)$ satisfying $P_k\subset\mathcal{P}_k(\mathbb{R}^n)$ and, for any $k\in\{0,\cdots,d\}$,

$$\int_B P_k(x)P_j(x)\,dx=\begin{cases}1 & \text{when } j=k,\\ 0 & \text{when } j\neq k.\end{cases}$$

This thesis have the unique decomposition of $P_B^d f$,

$$P_B^d f:=a_dP_d+\cdots+a_0P_0,$$

where $a_d,\dots,a_0\in\mathbb{R}$. Thus, this thesis has, for any $k\in\{0,\dots,d\}$,

$$\begin{aligned}a_k&=\int_B a_kP_k(x)P_k(x)\,dx=\int_B P_B^d(x)P_k(x)\,dx\\ &=\int_B f(x)P_k(x)\,dx\le\sup_{x\in B}|P_k|(x)\int_B|f(x)|\,dx.\end{aligned}$$

This further implies that, for any $x\in B$,

$$\begin{aligned}|P_B^df(x)|&\le\sum_{k=0}^d|a_k||P_k(x)|\le\sum_{k=0}^d[\sup_{x\in B}|P_k|(x)]^2\int_B|f(x)|\,dx\\ &\le\sum_{k=0}^d\frac{1}{|B|}\int_B|f(x)|\,dx=d\fint_B|f(x)|\,dx.\end{aligned}$$

By the arbitrary of x, this thesis complete the proof of Lemma 2.2.5. □

Now, this thesis proves that the dual space of $H_X^A(\mathbb{R}^n)$ is $\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$.

s2t1 **Theorem 2.2.6.** *Let A, X, q, d, and θ_0 be the same as in Definition 2.2.1. Further assume that X has an absolutely continuous quasi-norm. Then the dual space of $H_X^A(\mathbb{R}^n)$, denoted by $(H_X^A(\mathbb{R}^n))^*$, is $\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$ with $1/q+1/q'=1$ in the following sense:*

(i) *Let $g\in\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$. Then the linear functional*

s2t1e1 (2.2.4)
$$L_g:\ f\to L_g(f):=\int_{\mathbb{R}^n}f(x)g(x)\,dx,$$

initially defined for any $f\in H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)$, has a bounded extension to $H_X^A(\mathbb{R}^n)$.

(ii) *Conversely, any continuous linear functional on $H_X^A(\mathbb{R}^n)$ arises as in (2.2.4) with a unique $g\in\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$.*

Moreover, $\|g\|_{\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)}\sim\|L_g\|_{(H_X^A(\mathbb{R}^n))^*}$, where the positive equivalence constants are independent of g.

Proof. This thesis first show (i) in the case $q\in(\max\{1,p_0\},\infty)$ with p_0 the same as in Definition 2.2.1. To this end, let $g\in\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$. For any $f\in H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)$, by Definition 2.2.1, this thesis know that there exists a sequence $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$ and a sequence $\{a_j\}_{j=1}^m$ of anisotropic $(X,\ q,\ d)$-atoms supported, respectively, in the balls $\{B^{(j)}\}_{j=1}^m\subset\mathcal{B}$ such that $f=\sum_{j=1}^m\lambda_ja_j$ and

$$\left\|\left\{\sum_{j=1}^m\left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right]^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X\sim\|f\|_{H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)}.$$

From these, the vanishing moments of a_j, the Hölder inequality, the size condition of a_j, Remark 2.1.4(ii), Lemma 2.2.3(i), and Definition 2.1.3 with q replaced by q', s replaced by θ_0, it follows that

s2t1e2 (2.2.5)
$$\begin{aligned}|L_g(f)|&=\left|\int_{\mathbb{R}^n}f(x)g(x)\,dx\right|\le\sum_{j=1}^m\lambda_j\left|\int_{B^{(j)}}a_j(x)g(x)\,dx\right|\\&=\sum_{j=1}^m\lambda_j\inf_{P\in\mathcal{P}_d(\mathbb{R}^n)}\left|\int_{B^{(j)}}a_j(x)\left[g(x)-P(x)\right]\,dx\right|\\&\le\sum_{j=1}^m\lambda_j\|a_j\|_{L^q(\mathbb{R}^n)}\inf_{P\in\mathcal{P}_d(\mathbb{R}^n)}\left[\int_{B^{(j)}}|g(x)-P(x)|^{q'}\,dx\right]^{\frac{1}{q'}}\\&\le\sum_{j=1}^m\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\inf_{P\in\mathcal{P}_d(\mathbb{R}^n)}\left[\fint_{B^{(j)}}|g(x)-P(x)|^{q'}\,dx\right]^{\frac{1}{q'}}\end{aligned}$$

$$\lesssim \left\| \left\{ \sum_{i=1}^{m} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{\theta_0}} \right\|_X \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)}$$

$$\sim \|f\|_{H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)} \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \sim \|f\|_{H^A_X(\mathbb{R}^n)} \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)}.$$

Moreover, by [97, Lemma 7.2] and the assumption that X has an absolutely continuous quasi-norm, this thesis finds that $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ is dense in $H^A_X(\mathbb{R}^n)$. This, together with (2.2.5) and a standard density argument, further implies that, when $q \in (\max\{1, p_0\}, \infty)$, (i) holds true and

$$\|L_g\|_{(H^A_X(\mathbb{R}^n))^*} \lesssim \|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)}$$

with the implicit positive constant independent of g.

This thesis next prove (i) in the case $q = \infty$. Indeed, using Proposition 2.2.4 and repeating the above proof for any given $q \in (\max\{1, p_0\}, \infty)$, this thesis then conclude that any $g \in \mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ induces a bounded linear functional on $H^A_X(\mathbb{R}^n)$, which is initially defined on $H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$ and given by setting, for any $\ell \in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$,

s2t1e3 (2.2.6)
$$L_g: \ \ell \mapsto \ L_g(\ell) := \int_{\mathbb{R}^n} \ell(x) g(x)\, dx,$$

and then has a bounded linear extension to $H^A_X(\mathbb{R}^n)$. Let $g \in \mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$. Thus, it remains to show that, for any $f \in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$,

s2t1e4 (2.2.7)
$$L_g(f) = \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

To this end, suppose $f \in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ and $\operatorname{supp} f \subset x_0 + B_{i_0}$ with $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathbb{Z}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp} \varphi \subset B_0$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Letting $s \in (\max\{1, p_0\}, \infty)$, by the proof of Proposition 2.2.4, this thesis finds that, for any $k \in (-\infty, 0] \cap \mathbb{Z}$ and $f \in L^s(\mathbb{R}^n)$,

2.16.x1 (2.2.8)
$$\varphi_k * f \in H^{A,\infty,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$$

and

s2t1e5 (2.2.9)
$$\lim_{k \in (-\infty,0] \cap \mathbb{Z},\, k \to -\infty} \|f - \varphi_k * f\|_{L^s(\mathbb{R}^n)} = 0.$$

From this and the Riesz lemma (see, for instance, [35, Theorem 2.30]), it follows that there exists a subsequence $\{k_h\}_{h \in \mathbb{N}} \subset (-\infty, 0] \cap \mathbb{Z}$ such that $\lim_{h \to \infty} k_h = -\infty$ and, for almost every $x \in \mathbb{R}^n$,

$$\lim_{h \to \infty} \varphi_{k_h} * f(x) = f(x).$$

By (2.2.9) and an argument similar to that used in the proof of Proposition 2.2.4, this thesis concludes that $\lim_{h\to\infty}\|f-\varphi_{k_h}*f\|_{H_X^A(\mathbb{R}^n)}=0$, which, combined with Lemma 2.2.2, (2.2.8), (2.2.6), the fact that

$$|(\varphi_{k_h}*f)\,g|\le\|f\|_{L^\infty(\mathbb{R}^n)}\mathbf{1}_{x_0+B_{\max\{i_0,0\}+\tau}}|g|\in L^1(\mathbb{R}^n),$$

and the Lebesgue dominated convergence theorem (see, for instance, [35, Theorem 2.24]), further implies that

$$\begin{aligned}L_g(f)&=\lim_{h\to\infty}L_g(\varphi_{k_h}*f)=\lim_{h\to\infty}\int_{\mathbb{R}^n}\varphi_{k_h}*f(x)g(x)\,dx\\&=\int_{\mathbb{R}^n}f(x)g(x)\,dx.\end{aligned}$$

This finishes the proof of (2.2.7) and hence (i) in the case $q=\infty$. Moreover, repeating the proof in (2.2.5), this thesis obtains, for any $q\in(\max\{1,p_0\},\infty]$,

4.14.x2 (2.2.10)
$$\|L_g\|_{(H_X^A(\mathbb{R}^n))^*}\lesssim\|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)}$$

with the implicit positive constant independent of g.

This thesis next show (ii). For this purpose, let $\pi_B:\ L^1(B)\to\mathcal{P}_d(\mathbb{R}^n)$, with $B\in\mathcal{B}$, be the natural projection such that, for any $f\in L^1(B)$ and $Q\in\mathcal{P}_d(\mathbb{R}^n)$,

2.16.y3 (2.2.11)
$$\int_B\pi_B(f)(x)Q(x)\,dx=\int_B f(x)Q(x)\,dx.$$

For any $q\in(\max\{1,p_0\},\infty]$ and any ball $B\in\mathcal{B}$, the *closed subspace* $L_0^q(B)$ of $L^q(B)$ is defined by setting

$$L_0^q(B):=\{f\in L^q(B):\ \pi_B(f)=0\text{ and }f\neq0\text{ almost everywhere}\},$$

where $L^q(B)$ is the subspace of $L^q(\mathbb{R}^n)$ consisting of all the measurable functions on $\mathbb{R}^n$ vanishing outside B. For any $f\in L_0^q(B)$, since $f\neq0$ almost everywhere, this thesis can easily deduce $\|f\|_{L^q(\mathbb{R}^n)}\neq0$. Therefore,

$$\frac{|B|^{\frac1q}}{\|\mathbf{1}_B\|_X}\|f\|^{-1}_{L^q(\mathbb{R}^n)}f$$

is an anisotropic $(X,\ q,\ d)$-atom. From this and Lemma 2.2.2, it follows that

s2t1e6 (2.2.12)
$$\left\|\frac{|B|^{1/q}}{\|\mathbf{1}_B\|_X}\|f\|^{-1}_{L^q(\mathbb{R}^n)}f\right\|_{H_X^A(\mathbb{R}^n)}\lesssim1.$$

Now, suppose $L\in(H_X^A(\mathbb{R}^n))^*$. Then, by (2.2.12), this thesis finds that, for any $f\in L_0^q(B)$,

s2t1e7 (2.2.13)
$$|L(f)|\le\|L\|_{(H_X^A(\mathbb{R}^n))^*}\frac{\|\mathbf{1}_B\|_X}{|B|^{1/q}}\|f\|_{L^q(\mathbb{R}^n)}.$$

Therefore, L provides a bounded linear functional on $L^q_0(B)$. Thus, applying the Hahn–Banach theorem (see, for instance, [35, Theorem 5.6]), this thesis finds that there exists a linear functional L_B, which extends L to the whole space $L^q(B)$ without increasing its norm.

When $q \in (\max\{1, p_0\}, \infty)$, by the duality $(L^q(B))^* = L^{q'}(B)$, this thesis finds that there exists an $h_B \in L^{q'}(B) \subset L^1(B)$ such that, for any $f \in L^q_0(B)$,

s2t1e8

$$L(f) = L_B(f) = \int_B f(x) h_B(x)\, dx. \tag{2.2.14}$$

In the case $q = \infty$, let $\widetilde{q} \in (\max\{1, p_0\}, \infty)$. Then there exists an $h_B \in L^{\widetilde{q}'}(B) \subset L^1(B)$ such that, for any $f \in L^\infty_0(B) \subset L^{\widetilde{q}}(B)$, $L(f) = \int_B f(x) h_B(x)\, dx$. Altogether, this thesis finds that, for any $q \in (\max\{1, p_0\}, \infty]$, there exists an $h_B \in L^{q'}(B)$ such that, for any $f \in L^q_0(B)$,

2.16.y2

$$L(f) = \int_B f(x) h_B(x)\, dx. \tag{2.2.15}$$

Next, this thesis proves that such an $h_B \in L^{q'}(B)$ is unique in the sense of modulo $\mathcal{P}_d(\mathbb{R}^n)$. Indeed, assume that $\widetilde{h_B}$ is another element of $L^{q'}(B)$ such that

2.16.y1

$$L(f) = \int_B f(x) \widetilde{h_B}(x)\, dx \tag{2.2.16}$$

for any $f \in L^q_0(B)$. Then, from (2.2.15), (2.2.16), and (2.2.11), this thesis infers that, for any $f \in L^\infty(B)$, $f - \pi_B(f) \in L^\infty_0(B)$ and

$$\begin{aligned}
0 &= \int_B [f(x) - \pi_B(f)(x)] \left[h_B(x) - \widetilde{h_B}(x)\right] dx \\
&= \int_B f(x) \left[h_B(x) - \widetilde{h_B}(x)\right] dx - \int_B \pi_B(f)(x) \pi_B(h_B - \widetilde{h_B})(x)\, dx \\
&= \int_B f(x) \left[h_B(x) - \widetilde{h_B}(x)\right] dx - \int_B f(x) \pi_B(h_B - \widetilde{h_B})(x)\, dx \\
&= \int_B f(x) \left[h_B(x) - \widetilde{h_B}(x) - \pi_B(h_B - \widetilde{h_B})(x)\right] dx.
\end{aligned}$$

The arbitrariness of f further implies that $h_B(x) - \widetilde{h_B}(x) = \pi_B(h_B - \widetilde{h_B})(x)$ for almost every $x \in B$. Therefore, after changing values of h_B (or $\widetilde{h_B}$) on a set of measure zero, this thesis has $h_B - \widetilde{h_B} \in \mathcal{P}_d(\mathbb{R}^n)$. Strictly speaking, since this thesis are dealing with uncountably many balls B, the change of the value of h_B must be done carefully. However, since this is just a matter of passing to the limit starting from a countable family of balls. So, we ignore this issue. Thus, for any $q \in (\max\{1, p_0\}, \infty]$ and $f \in L^q_0(B)$, there exists a unique $h_B \in L^{q'}(B)/\mathcal{P}_d(B)$ such that (2.2.14) holds true.

For any $j \in \mathbb{R}^n$ and $f \in L_0^q(B_j)$, let g_j be the unique element of $L^{q'}(B_j)/\mathcal{P}_d(B_j)$ such that

$$L(f) = \int_{B_j} f(x) g_j(x)\, dx.$$

Therefore, this thesis can define a local $L^{q'}(\mathbb{R}^n)$ function g by setting $g(x) := g_j(x)$ whenever $x \in B_j$. Assume that f is a finite linear combination of anisotropic $(X,\ q,\ d)$-atoms. It is easy to show that there exists an $x_0 \in \mathbb{R}^n$ and a $k_0 \in \mathbb{Z}$ such that $\operatorname{supp} f \subset x_0 + B_{k_0}$. Let

$$j_0 := \frac{\ln A_0 + \ln[b^{k_0-1} + \rho(x_0)]}{\ln b} + 1.$$

Then, by Definition 1.2.3, this thesis concludes that $\operatorname{supp} f \subset x_0 + B_{k_0} \subset B_{j_0}$. Thus, $f \in L_0^q(B_{j_0})$ and

$$L(f) = \int_{B_{j_0}} f(x) g_{j_0}(x)\, dx = \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

From this and (2.2.13), this thesis deduces that, for any ball $B \in \mathcal{B}$,

s2t1e9
$$\|g\|_{(L_0^q(B))^*} \le \frac{\|\mathbf{1}_B\|_X}{|B|^{1/q}} \|L\|_{(H_X^A(\mathbb{R}^n))^*}. \tag{2.2.17}$$

Moreover, it is known that

$$\|g\|_{(L_0^q(B))^*} = \inf_{P \in \mathcal{P}_d(\mathbb{R}^n)} \|g - P\|_{L^{q'}(B)}$$

(see, for instance, [4, p. 52, (8.12)]), which, combined with Remark 2.1.7(ii) and (2.2.17), further implies that

4.14
$$\|g\|_{\mathcal{L}^A_{X,q',d}(\mathbb{R}^n)} \sim \sup_{B \in \mathcal{B}} \frac{|B|^{\frac{1}{q}}}{\|\mathbf{1}_B\|_X} \|g\|_{(L_0^q(B))^*} \le \|L\|_{(H_X^A(\mathbb{R}^n))^*}. \tag{2.2.18}$$

Thus, $g \in \mathcal{L}^A_{X,q',d}(\mathbb{R}^n)$ and, for any finite linear combination f of anisotropic $(X,\ q,\ d)$-atoms,

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x)\, dx.$$

Now, this thesis shows that $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$ and $\|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \lesssim \|L\|_{(H_X^A(\mathbb{R}^n))^*}$. To this end, for any $m \in \mathbb{N}$, $\{B^{(j)}\}_{j=1}^m \subset \mathcal{B}$, and $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$, let $h_j \in L^q(B^{(j)})$ with $\|h_j\|_{L^q(B^{(j)})} = 1$ be such that

s2t1e10
$$\begin{aligned} &\left[\int_{B^{(j)}} \left| g(x) - P^d_{B^{(j)}} g(x) \right|^{q'} dx \right]^{\frac{1}{q'}} \\ &\quad = \int_{B^{(j)}} \left[g(x) - P^d_{B^{(j)}} g(x) \right] h_j(x)\, dx \end{aligned} \tag{2.2.19}$$

and, for any $x \in \mathbb{R}^n$, define

$$a_j(x) := \frac{|B^{(j)}|^{\frac{1}{q}}[h_j(x) - P^d_{B^{(j)}} h_j(x)]\mathbf{1}_{B^{(j)}}(x)}{\|\mathbf{1}_{B^{(j)}}\|_X \|h_j - P^d_{B^{(j)}} h_j\|_{L^q(B^{(j)})}}.$$

Then it is easy to find that, for any $j \in \{1, \ldots, m\}$, a_j is an anisotropic $(X,\ q,\ d)$-atom. From this, and Lemma 2.2.2, it follows that $\sum_{j=1}^m \lambda_j a_j \in H^A_X(\mathbb{R}^n)$ and

h_j

$$\left\| \sum_{j=1}^m \lambda_j a_j \right\|_{H^A_X(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j=1}^m \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X . \tag{2.2.20}$$

Moreover, by the Minkowski inequality, the assumption that $\|h_j\|_{L^q(B^{(j)})} = 1$, Lemma 2.2.5, and the Hölder inequality, this thesis finds that

$$\begin{aligned}
\left\| h_j - P^d_{B^{(j)}} h_j \right\|_{L^q(B^{(j)})} &\le \|h_j\|_{L^q(B^{(j)})} + \left\| P^d_{B^{(j)}} h_j \right\|_{L^q(B^{(j)})} \\
&\lesssim 1 + \left| B^{(j)} \right|^{\frac{1}{q}} \fint_{B^{(j)}} |h_j(x)|\, dx \\
&= 1 + \frac{1}{|B^{(j)}|^{\frac{1}{q'}}} \int_{B^{(j)}} |h_j(x)|\, dx \\
&\le 1 + \|h_j\|_{L^q(B^{(j)})} \lesssim 1.
\end{aligned}$$

This, together with (2.2.19), the assumption that $L \in (H^A_X(\mathbb{R}^n))^*$, and (2.2.20), further implies that

$$\begin{aligned}
&\sum_{j=1}^m \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[\fint_{B^{(j)}} \left| g(x) - P^d_{B^{(j)}} g(x) \right|^{q'} dx \right]^{\frac{1}{q'}} \\
&\quad = \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(j)}}\|_X} \int_{B^{(j)}} \left[g(x) - P^d_{B^{(j)}} g(x) \right] h_j(x)\, dx \\
&\quad = \sum_{j=1}^m \frac{\lambda_j |B^{(j)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(j)}}\|_X} \int_{B^{(j)}} \left[h_j(x) - P^d_{B^{(j)}} h_j(x) \right] g(x) \mathbf{1}_{B^{(j)}}(x)\, dx \\
&\quad \lesssim \sum_{j=1}^m \lambda_j \int_{B^{(j)}} a_j(x) g(x)\, dx = \sum_{j=1}^m \lambda_j L(a_j) = L\left(\sum_{j=1}^m \lambda_j a_j \right) \\
&\quad \lesssim \left\| \sum_{j=1}^m \lambda_j a_j \right\|_{H^A_X(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{j=1}^m \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X .
\end{aligned}$$

Using this and Definition 2.1.3, this thesis finds $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$. Moreover, from $g \in \mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$, Proposition 2.1.9, and (2.2.18), this thesis infers that

$$\|g\|_{\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)} \sim \|g\|_{\mathcal{L}^A_{X,q',d}(\mathbb{R}^n)} \lesssim \|L\|_{(H^A_X(\mathbb{R}^n))^*}.$$

This finishes the proof of (ii) and hence Theorem 2.2.6. □

As a consequence of Theorem 2.2.6, this thesis has the following equivalence of the anisotropic ball Campanato function space $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$; this thesis omits the details.

s2c1 **Corollary 2.2.7.** *Let A, X, d, θ_0, and p_0 be the same as in Theorem 2.2.6 and $q \in [1,\infty)$ when $p_0 \in (0,1)$, or $q \in [1, p_0')$ when $p_0 \in [1,\infty)$. Then*

$$\mathcal{L}^A_{X,1,d_{X,A},\theta_0}(\mathbb{R}^n) = \mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$$

with equivalent quasi-norms, where $d_{X,A}$ is the same as in (2.2.2).

3.24.x1 **Remark 2.2.8.** (i) If $A := 2\, I_{n\times n}$, then Theorem 2.2.6 and Corollary 2.2.7 were obtained in [103, Theorem 3.14 and Corollary 3.15], respectively.

(ii) Recently, Yan et al. [99, Theorem 6.6] obtained the dual theorem of the Hardy space $H_Y(\mathcal{X})$ associated with the ball quasi-Banach function space $Y(\mathcal{X})$ on a given space $\mathcal{X}$ of homogeneous type. This thesis point out that, since there exists no linear structure in a general space $\mathcal{X}$ of homogeneous type, one can not introduce the Schwartz function and the polynomial on $\mathcal{X}$. Indeed, any atom in [99] only has zero degree vanishing moment, while the atom in Theorem 2.2.6 has vanishing moments up to order $d \in [d_{X,\,A}, \infty) \cap \mathbb{N}$ with $d_{X,\,A}$ the same as in (2.2.2). Thus, although $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type, Theorem 2.2.6 can not be deduced from by [99, Theorem 6.6] and, actually, they can not cover each other.

Chapter 3

c3 Real-Variable Characterizations of $H^A_X(\mathbb{R}^n)$

c3s1 3.1 Littlewood–Paley Function Characterizations of $H^A_X(\mathbb{R}^n)$

In this section, this thesis establishes the characterizations of $H^A_X(\mathbb{R}^n)$ in terms of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, or the anisotropic Littlewood–Paley g^*_λ-function. These are the consequence of the atomic and the finite atomic characterizations of $H^A_X(\mathbb{R}^n)$ obtained in [97] and play important roles in establishing the Carleson measure characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ in Section 4.2. First, this thesis recall the concepts of both the anisotropic radial maximal function and the anisotropic radial grand maximal function, which were introduced in [4].

radialM **Definition 3.1.1.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *anisotropic radial maximal function* $M^0_\varphi(f)$ of f with respect to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M^0_\varphi(f)(x) := \sup_{k\in\mathbb{Z}} |f * \varphi_k(x)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *anisotropic radial grand maximal function* $M^0_N(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M^0_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M^0_\varphi(f)(x).$$

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\varphi}$ is defined by setting, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi\imath x\cdot\xi}\,dx,$$

where $\imath := \sqrt{-1}$ and $x \cdot \xi := \sum_{i=1}^n x_i\xi_i$ for any $x := (x_1, \ldots, x_n), \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ is defined by setting, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{f}, \varphi\rangle := \langle f, \widehat{\varphi}\rangle$.

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$ (see, for instance, [36, p. 50]). Let $\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ denote the collection of all the infinitely differentiable functions with compact support on $\mathbb{R}^n$. The following Calderón reproducing formula is just [7, Proposition 2.14].

s4l1 **Lemma 3.1.2.** *Let $d \in \mathbb{Z}_+$ and A be a dilation. Assume that $\phi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ satisfies*

s4l1e1 (3.1.1)
$$\operatorname{supp} \phi \subset B_0, \ \int_{\mathbb{R}^n} x^{\gamma} \phi(x)\, dx = 0 \ \textit{for any } \gamma \in \mathbb{Z}_+^n \textit{ with } |\gamma| \le d,$$

and there exists a positive constant C such that

s4l1e2 (3.1.2)
$$\left|\widehat{\phi}(\xi)\right| \ge C \textit{ when } \xi \in \left\{x \in \mathbb{R}^n : \ (2\|A\|)^{-1} \le \rho(x) \le 1\right\},$$

Then there exists a $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

(i) supp $\widehat{\psi}$ *is compact and away from the origin;*

(ii) *for any $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,*
$$\sum_{j \in \mathbf{Z}} \widehat{\psi}\left((A^*)^j \xi\right) \widehat{\phi}\left((A^*)^j \xi\right) = 1,$$
where A^* denotes the adjoint matrix of A.

Moreover, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, if f vanishes weakly at infinity, then
$$f = \sum_{j \in \mathbb{Z}} f * \psi_j * \phi_j \ \textit{in } \mathcal{S}'(\mathbb{R}^n).$$

The following definitions of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g_λ^*-function were introduced in [76, Definition 2.6].

de4.1 **Definition 3.1.3.** Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be the same as in Lemma 3.1.2. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *anisotropic Lusin area function* $S(f)$, the *anisotropic Littlewood–Paley g-function* $g(f)$, and the *anisotropic Littlewood–Paley g_λ^*-function* $g_\lambda^*(f)$ with any given $\lambda \in (0, \infty)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

deaf (3.1.3)
$$S(f)(x) := \left[\sum_{k \in \mathbb{Z}} b^{-k} \int_{x + B_k} |f * \phi_k(y)|^2 \, dy\right]^{\frac{1}{2}},$$

$$g(f)(x) := \left[\sum_{k \in \mathbb{Z}} |f * \phi_k(x)|^2\right]^{\frac{1}{2}},$$

and
$$g_\lambda^*(f)(x) := \left\{\sum_{k \in \mathbb{Z}} b^{-k} \int_{\mathbb{R}^n} \left[\frac{b^k}{b^k + \rho(x - y)}\right]^{\lambda} |f * \phi_k(y)|^2 \, dy\right\}^{\frac{1}{2}}.$$

This thesis characterize the space $H_X^A(\mathbb{R}^n)$, respectively, in terms of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g_λ^*-function as follows.

s4t1 **Theorem 3.1.4.** *Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 1.2.10 with $p_- \in (0,\infty)$ and Assumption 1.2.12 with the same p_-, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\theta_0, \infty)$, where $\underline{p}$ is the same as in* (1.2.6)*. Then $f \in H_X^A(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $\|S(f)\|_X < \infty$. Moreover, for any $f \in H_X^A(\mathbb{R}^n)$,*

$$\|S(f)\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

s4t1' **Theorem 3.1.5.** *Let A and X be the same as in Theorem 3.1.4. Then $f \in H_X^A(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $\|g(f)\|_X < \infty$. Moreover, for any $f \in H_X^A(\mathbb{R}^n)$,*

$$\|g(f)\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f.

Moreover, by Theorems 3.1.4 and 3.1.5 and an argument similar to that used in the proof of [17, Theorem 4.11], this thesis easily obtain the following result; this thesis omit the details.

s4t1'' **Theorem 3.1.6.** *Let A, X, and θ_0 be the same as in Theorem 3.1.4, $\lambda \in (\max\{1, 2/r_+\}, \infty)$, where*

3.25.x1
$$r_+ := \sup\{\theta_0 \in (0,\infty) : X \text{ satisfies Assumption 1.2.12 for this } \theta_0 \text{ and some } p_0 \in (\theta_0, \infty)\}. \tag{3.1.4}$$

Then $f \in H_X^A(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $\|g_\lambda^*(f)\|_X < \infty$. Moreover, for any $f \in H_X^A(\mathbb{R}^n)$,

$$\|g_\lambda^*(f)\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)},$$

where the positive equivalence constants are independent of f,

To prove Theorem 3.1.4, this thesis first present the following conclusion which shows that the quasi-norm $\|\cdot\|_X$ of the anisotropic Lusin area functions defined by different ϕ as in Lemma 3.1.2 are equivalent.

s4t2 **Theorem 3.1.7.** *Let A and X be the same as in Theorem 3.1.4 and $\phi, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfy both* (3.1.1) *and* (3.1.2)*. Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity,*

$$\|S_\phi(f)\|_X \sim \|S_\psi(f)\|_X,$$

where $S_\phi(f)$ is the same as in (3.1.3)*, $S_\psi(f)$ is the same as in* (3.1.3) *with ϕ replaced by ψ, and the positive equivalence constants are independent of f.*

To prove Theorem 3.1.7, this thesis needs the following lemma which is just [7, Lemma2.3] and originates from [19, Theorem 11].

defdya **Lemma 3.1.8.** *Let A be a dilation. Then there exists a collection*

$$\mathcal{Q} := \left\{Q_\alpha^k \subset \mathbb{R}^n : \ k \in \mathbb{Z}, \alpha \in I_k\right\}$$

of open subsets, where I_k is certain index set, such that

(i) $|\mathbb{R}^n \setminus \bigcup_\alpha Q_\alpha^k| = 0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ for any $\alpha \neq \beta$;*

(ii) *for any α, β, k, ℓ with $\ell \geq k$, either $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ or $Q_\alpha^\ell \subset Q_\beta^k$;*

(iii) *for each (ℓ, β) and each $k < \ell$, there exists a unique α such that $Q_\beta^\ell \subset Q_\alpha^k$;*

(iv) *there exist certain negative integer v and positive integer u such that, for any Q_α^k with both $k \in \mathbb{Z}$ and $\alpha \in I_k$, there exists an $x_{Q_\alpha^k} \in Q_\alpha^k$ satisfying that, for any $x \in Q_\alpha^k$,*

$$x_{Q_\alpha^k} + B_{vk-u} \subset Q_\alpha^k \subset x + B_{vk+u}.$$

In what follows, for convenience, this thesis call $\mathcal{Q} := \{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in I_k}$ in Lemma 3.1.8 *dyadic cubes* and k the *level*, denoted by $\ell(Q_\alpha^k)$, of the dyadic cube Q_α^k with both $k \in \mathbb{Z}$ and $\alpha \in I_k$.

The following technical lemma is also necessary, which is just [47, Lemma 6.9].

s412 **Lemma 3.1.9.** *Let d be the same as in (2.2.1), v and u the same as in Lemma 3.1.8(iv), and*

$$\eta \in \left(\frac{\ln b}{\ln b + (d+1)\ln \lambda_-}, 1\right].$$

Then there exists a positive constant C such that, for any $k, i \in \mathbb{Z}$, $\{c_Q\}_{Q\in\mathcal{Q}} \subset [0,\infty)$ with $\mathcal{Q}$ in Lemma 3.1.8, and $x \in \mathbb{R}^n$,

$$\begin{aligned}&\sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} |Q| \frac{b^{(k\vee i)(d+1)\frac{\ln\lambda_-}{\ln b}}}{[b^{(k\vee i)} + \rho(x - z_Q)]^{(d+1)\frac{\ln\lambda_-}{\ln b}+1}} c_Q \\ &\quad \leq C b^{-[k-(k\vee i)](\frac{1}{\eta}-1)} \left\{\mathcal{M}\left[\sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} (c_Q)^\eta \mathbf{1}_Q\right](x)\right\}^{\frac{1}{\eta}},\end{aligned}$$

where $\ell(Q)$ denotes the level of Q, $z_Q \in Q$, and, for any $k, i \in \mathbb{Z}$, $k \vee i :=$ $\max\{k, i\}$.

This thesis now prove Theorem 3.1.7.

Proof of Theorem 3.1.7. By symmetry, to show the present theorem, this thesis only need to prove that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ which vanishes weakly at infinity,

s4e1 (3.1.5)
$$\|S_\phi(f)\|_X \lesssim \|S_\psi(f)\|_X \,.$$

To this end, for any $i \in \mathbb{Z}, x \in \mathbb{R}^n$, and $y \in x + B_i$, let

$$J_\phi^{(i)}(f)(y) := f * \phi_i(y).$$

Then, by Lemma 3.1.2 and the Lebesgue dominated convergence theorem, this thesis finds that, for any $i \in \mathbb{Z}, x \in \mathbb{R}^n$, and $y \in x + B_i$,

s4e2 (3.1.6)
$$\begin{aligned} J_\phi^{(i)}(f)(y) &= \sum_{k\in\mathbb{Z}} f * \psi_k * \phi_k * \phi_i(y) \\ &= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} f * \psi_k(z) \phi_k * \phi_i(y-z)\, dz \\ &= \sum_{k\in\mathbb{Z}} \sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} \int_Q f * \psi_k(z) \phi_k * \phi_i(y-z)\, dz \end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where all the symbols are the same as in Lemma 3.1.9. On the other hand, by [8, Lemma 5.4], this thesis concludes that, for any $k, i \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|\phi_k * \phi_i(x)| \lesssim b^{-(d+1)|k-i|\frac{\ln \lambda_-}{\ln b}} \frac{b^{(k\vee i)(d+1)\frac{\ln \lambda_-}{\ln b}}}{\left[b^{(k\vee i)} + \rho(x)\right]^{(d+1)\frac{\ln \lambda_-}{\ln b}+1}}.$$

This further implies that, for any $Q \in \mathcal{Q}$ with

s4e3 (3.1.7)
$$\ell(Q) = \left\lceil \frac{k-u}{v} \right\rceil,$$

there exists some $z_Q \in Q$ such that, for any $k, i \in \mathbb{Z}, x \in \mathbb{R}^n, y \in x + B_i$, and $z \in Q$,

s4e4 (3.1.8)
$$|\phi_k * \phi_i(y-z)| \lesssim b^{-(d+1)|k-i|\frac{\ln \lambda_-}{\ln b}} \frac{b^{(k\vee i)(d+1)\frac{\ln \lambda_-}{\ln b}}}{\left[b^{(k\vee i)} + \rho\left(x - z_Q\right)\right]^{(d+1)\frac{\ln \lambda_-}{\ln b}+1}}.$$

Moreover, for any $Q \in \mathcal{Q}$ satisfying (3.1.7), this thesis has $B_{v\ell(Q)+u} \subset B_k$. From this, the Hölder inequality, and Lemma 3.1.8(iv), this thesis deduces that, for any $z \in Q$,

$$\begin{aligned} \frac{1}{|Q|}\left|\int_Q f * \psi_k(y)\, dy\right| &\le \left[⨍_Q |f * \psi_k(y)|^2\, dy\right]^{\frac{1}{2}} \\ &\le \left[\frac{1}{|B_{v\ell(Q)-u}|} \int_{z+B_{v\ell(Q)+u}} |f * \psi_k(y)|^2\, dy\right]^{\frac{1}{2}} \end{aligned}$$

$$\lesssim \left[b^{-k} \int_{z+B_k} |f * \psi_k(y)|^2 \, dy\right]^{\frac{1}{2}} \sim Y_\psi^{(k)}(f)(z),$$

where, for any $k \in \mathbb{Z}$ and $z \in \mathbb{R}^n$,

$$Y_\psi^{(k)}(f)(z) := \left[b^{-k} \int_{z+B_k} |f * \psi_k(y)|^2 \, dy\right]^{\frac{1}{2}}.$$

Thus, for any $k \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ satisfying (3.1.7),

$$\frac{1}{|Q|} \left| \int_Q f * \psi_k(y) \, dy \right| \lesssim \inf_{z \in Q} Y_\psi^{(k)}(f)(z).$$

By this, (3.1.6), (3.1.8), and Lemma 3.1.9, this thesis concludes that, for any given $\eta \in \left(\frac{\ln b}{\ln b + (d+1) \ln \lambda_-}, 1\right]$ and for any $i \in \mathbb{Z}, x \in \mathbb{R}^n$, and $y \in x + B_i$,

eqJei (3.1.9)

$$\begin{aligned}
\left|J_\phi^{(i)}(f)(y)\right| &\lesssim \sum_{k \in \mathbb{Z}} b^{-(d+1)|k-i| \frac{\ln \lambda_-}{\ln b}} \\
&\quad \times \sum_{\ell(Q) = \lceil \frac{k-u}{v} \rceil} |Q| \frac{b^{(k \vee i)(d+1) \frac{\ln \lambda_-}{\ln b}}}{[b^{(k \vee i)} + \rho(x - z_Q)]^{(d+1) \frac{\ln \lambda_-}{\ln b} + 1}} \inf_{z \in Q} Y_\psi^{(k)}(f)(z) \\
&\lesssim \sum_{k \in \mathbb{Z}} b^{-(d+1)|k-i| \frac{\ln \lambda_-}{\ln b}} b^{-[k-(k \vee i)](\frac{1}{\eta} - 1)} \\
&\quad \times \left\{ \mathcal{M} \left(\sum_{\ell(Q) = \lceil \frac{k-u}{v} \rceil} \inf_{z \in Q} \left[Y_\psi^{(k)}(f)(z) \right]^\eta \mathbf{1}_Q \right)(x) \right\}^{\frac{1}{\eta}} \\
&=: J_{(\eta, i)}(x).
\end{aligned}$$

Using (2.2.1), this thesis are able to choose an $\eta \in \left(\frac{\ln b}{\ln b + (d+1) \ln \lambda_-}, \theta_0\right)$. Therefore, from (3.1.9), it follows that, for such an η and any $x \in \mathbb{R}^n$,

$$[S_\phi(f)(x)]^2 = \sum_{i \in \mathbb{Z}} b^{-i} \int_{x+B_i} \left| J_\phi^{(i)}(f)(y) \right|^2 dy \lesssim \sum_{i \in \mathbb{Z}} \left[J_{(\eta, i)}(x) \right]^2.$$

This, together with the Hölder inequality and the choice that $\eta > \frac{\ln b}{\ln b + (d+1) \ln \lambda_-}$, further implies that, for such an η and any $x \in \mathbb{R}^n$,

$$\begin{aligned}
[S_\phi(f)(x)]^2 &\lesssim \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left\{ b^{-(d+1)|k-i| \frac{\ln \lambda_-}{\ln b}} b^{-[k-(k \vee i)](\frac{1}{\eta} - 1)} \right\}^2 \\
&\quad \times \sum_{k \in \mathbb{Z}} \left\{ \mathcal{M} \left(\sum_{\ell(Q) = \lceil \frac{k-u}{v} \rceil} \inf_{z \in Q} \left[Y_\psi^{(k)}(f)(z) \right]^\eta \mathbf{1}_Q \right)(x) \right\}^{\frac{2}{\eta}}
\end{aligned}$$

$$\lesssim \sum_{k\in\mathbb{Z}} \left\{ \mathcal{M}\left(\sum_{\ell(Q)=\lceil \frac{k-u}{v} \rceil} \inf_{z\in Q} \left[Y_{\psi}^{(k)}(f)(z) \right]^{\eta} \mathbf{1}_Q \right)(x) \right\}^{\frac{2}{\eta}}$$
$$\leq \sum_{k\in\mathbb{Z}} \left\{ \mathcal{M}\left(\left[Y_{\psi}^{(k)}(f) \right]^{\eta} \right)(x) \right\}^{\frac{2}{\eta}}.$$

Thus, by the fact that $\eta < \theta_0$ and Assumption 1.2.10, this thesis finds that

$$\|S_{\phi}(f)\|_X \lesssim \left\| \left(\sum_{k\in\mathbb{Z}} \left\{ \mathcal{M}\left(\left[Y_{\psi}^{(k)}(f) \right]^{\eta} \right)(x) \right\}^{\frac{2}{\eta}} \right)^{\frac{\eta}{2}} \right\|_{X^{\frac{1}{\eta}}}^{\frac{1}{\eta}}$$
$$\lesssim \left\| \left(\sum_{k\in\mathbb{Z}} \left[Y_{\psi}^{(k)}(f) \right]^{2} \right)^{\frac{1}{2}} \right\|_{X} = \|S_{\psi}(f)\|_X,$$

which further implies that (3.1.5) holds true and hence completes the proof of Theorem 3.1.7. □

Now, this thesis recall the concept of the anisotropic weight class of Muckenhoupt, associated with a dilation A, which was introduced in [6, Definition 2.4].

Definition 3.1.10. Let A be a dilation, $p \in [1, \infty)$, and w be a nonnegative measurable function on $\mathbb{R}^n$. The function w is said to belong to the *anisotropic weight class of Muckenhoupt*, $\mathcal{A}_p(A) := \mathcal{A}_p(\mathbb{R}^n, A)$, if there exists a positive constant C such that, when $p \in (1, \infty)$,

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left\{ \fint_{x+B_k} w(y)\, dy \right\} \left\{ \fint_{x+B_k} [w(y)]^{-\frac{1}{p-1}}\, dy \right\}^{p-1} \leq C$$

or, when $p = 1$,

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \left\{ \fint_{x+B_k} w(y)\, dy \right\} \left\{ \operatorname{ess} \sup_{y\in x+B_k} [w(y)]^{-1} \right\} \leq C.$$

Moreover, the minimal constants C as above are denoted by $C_{p,A,n}(w)$.

It is easy to prove that, if $1 \leq p \leq q \leq \infty$, then $\mathcal{A}_p(A) \subset \mathcal{A}_q(A)$. Let

$$\mathcal{A}_{\infty}(A) := \bigcup_{q\in[1,\infty)} \mathcal{A}_q(A).$$

For any given $w \in \mathcal{A}_{\infty}(A)$, define the *critical index* q_w of w by setting

qw
$$q_w := \inf\{p \in [1, \infty) :\ w \in \mathcal{A}_p(A)\}. \tag{3.1.10}$$

Obviously, $q_w \in [1,\infty)$. By the reverse Hölder inequality (see, for instance, [50, Theorem 1.2]), this thesis concludes that, for any $p \in (1,\infty)$ and $w \in \mathcal{A}_p(A)$, there exists an $\epsilon \in (0, p-1]$ such that $w \in \mathcal{A}_{p-\epsilon}(A)$. Thus, if $q_w \in (1,\infty)$, then $w \notin \mathcal{A}_{q_w}(A)$. Moreover, Johnson and Neugebauer [54, p. 254] gave an example of $w \notin \mathcal{A}_1(A)$ with $A = 2I_{n\times n}$ such that $q_w = 1$.

In what follows, for any nonnegative local integrable function w and any Lebesgue measurable set E, let

$$w(E) := \int_E w(x)\,dx.$$

For any given $p \in (0,\infty)$, denote by $L^p_w(\mathbb{R}^n)$ the *set of all the measurable functions* f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right\}^{\frac{1}{p}} < \infty.$$

Moreover, let $L^\infty_w(\mathbb{R}^n) := L^\infty(\mathbb{R}^n)$. Obviously, $L^p_w(\mathbb{R}^n)$ is a ball quasi-Banach function space, which even may not be a quasi-Banach function space (see, for instance, [81, p. 86]).

To show Theorem 3.1.4, this thesis needs the following several technical lemmas. Lemma 3.1.11 is a direct corollary of [99, Lemma 4.9] (see also [82, (4.6)]) because $(\mathbb{R}^n, \rho, dx)$ is a special space of homogeneous type; Lemma 3.1.12 is similar to [4, p. 21, Theorem 4.5] and this thesis omit the details.

embed **Lemma 3.1.11.** *Let A, X, and θ_0 be the same as in Theorem 3.1.4. Assume that $x_0 \in \mathbb{R}^n$. Then there exists an $\epsilon \in (0,1)$ such that X continuously embeds into $L^{\theta_0}_w(\mathbb{R}^n)$, where $w := [\mathcal{M}(\mathbf{1}_{x_0+B_0})]^\epsilon$ and B_0 is the same as in* (1.2.2) *with $k = 0$.*

inclu **Lemma 3.1.12.** *Let A and X be the same as in Theorem 3.1.4. Then $H^A_X(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous.*

Combining Lemmas 3.1.11 and 3.1.12, this thesis obtains the following property of $H^A_X(\mathbb{R}^n)$.

HXAvanish **Lemma 3.1.13.** *Let A and X be the same as in Theorem 3.1.4 and $f \in H^A_X(\mathbb{R}^n)$. Then f vanishes weakly at infinity.*

Proof. Let $N \in \mathbb{N}$ be the same as in (2.1.1). By Lemma 3.1.12, this thesis finds that, for any $k \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $y \in x + B_k$, $|f * \varphi_k(x)| \lesssim M_N(f)(y)$. Thus, there exists a positive constant C_1 such that, for any $k \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$x + B_k \subset \{y \in \mathbb{R}^n :\ M_N(f)(y) > C_1|f * \varphi_k(x)|\}.$$

By this, Lemma 3.1.11, and $[\mathcal{M}(\mathbf{1}_{x_0+B_0})]^\epsilon$ is not integrable on $\mathbb{R}^n$, this thesis concludes that, for any $k \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$|f * \varphi_k(x)| = [w(B_k)]^{-\frac{1}{\theta_0}}[w(B_k)]^{\frac{1}{\theta_0}}|f * \varphi_k(x)|$$

$$
\begin{aligned}
&\leq [w(B_k)]^{-\frac{1}{\theta_0}} \left[w\left(\{y \in \mathbb{R}^n : \ M_N(f)(y) > C_1|f * \varphi_k(x)|\}\right)\right]^{\frac{1}{\theta_0}} \\
&\quad \times |f * \varphi_k(x)| \\
&\lesssim [w(B_k)]^{-\frac{1}{\theta_0}} \|M_N(f)\|_{L^{\theta_0}_w(\mathbb{R}^n)} \lesssim [w(B_k)]^{-\frac{1}{\theta_0}} \|M_N(f)\|_X \\
&= [w(B_k)]^{-\frac{1}{\theta_0}} ||f||_{H^A_X(\mathbb{R}^n)} \to 0
\end{aligned}
$$

as $k \to \infty$, which further implies that f vanishes weakly at infinity. This finishes the proof of Lemma 3.1.13. □

To show Theorem 3.1.4, this thesis also need the following lemma whose proof is similar to that of [69, Lemma 4.2]; this thesis omit the details here.

s413 **Lemma 3.1.14.** *Let A, X, θ_0, and p_0 be the same as in Theorem 3.1.4, $q \in (\max\{p_0, 1\}, \infty]$, $k_0 \in \mathbb{Z}$, and $\varepsilon \in (0, \infty)$. Assume that $\{\lambda_i\}_{i\in\mathbb{N}} \subset [0, \infty)$, $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$, and $\{m_i^{(\varepsilon)}\}_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ satisfy that, for any $\varepsilon \in (0, \infty)$ and $i \in \mathbb{N}$,*

$$
\operatorname{supp} m_i^{(\varepsilon)} := \left\{x \in \mathbb{R}^n : \ m_i^{(\varepsilon)} \neq 0\right\} \subset A^{k_0} B^{(i)},
$$

$$
\|m_i^{(\varepsilon)}\|_{L^q(\mathbb{R}^n)} \leq \frac{|B^{(i)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(i)}}\|_X},
$$

and

$$
\left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{||\mathbf{1}_{B^{(i)}}||_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X < \infty.
$$

Then

$$
\left\| \liminf_{\varepsilon\to 0^+} \left[\sum_{i\in\mathbb{N}} \left|\lambda_i m_i^{(\varepsilon)}\right|^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X \leq C \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{||\mathbf{1}_{B^{(i)}}||_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X,
$$

where C is a positive constant independent of λ_i, $B^{(i)}$, $m_i^{(\varepsilon)}$, and ε.

Now, this thesis proves Theorem 3.1.4.

Proof of Theorem 3.1.4. Let τ be the same as in (1.2.4) and u and v the same as in Lemma 3.1.8(iv). This thesis first show the necessity of the present theorem. To this end, let $f \in H^A_X(\mathbb{R}^n)$. Then, by Lemma 3.1.13, this thesis finds that f vanishes weakly at infinity. On the other hand, it follows from [97, Theorem 4.3] that there exists a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset [0, \infty)$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of anisotropic (X, q, d)-atoms supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$ such that

$$
f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n)
$$

and

$$\|f\|_{H_X^A(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X .$$

Let a be an (X, q, d)-atom supported in a dyadic cube Q. Let $w := u - v + 2\tau$ and, for any $j \in \mathbb{N}$, $U_j := x_Q + (B_{v[\ell(Q)-j-1]+2\tau} \setminus B_{v[\ell(Q)-j]+2\tau})$. Then, by Lemma 3.1.8(iv), this thesis concludes that, for any $x \in (A^w Q)^\complement$, there exists some $j_0 \in \mathbb{N}$ such that $x \in U_{j_0}$. For this j_0, choose an $N \in \mathbb{N}$ lager enough such that

$$(N - \beta)vj_0 + \left(\frac{1}{q} - \beta\right) u < 0,$$

where $\beta := \left(\frac{\ln b}{\ln \lambda_-} + d + 1\right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{\theta_0}$. By this and an argument similar to that used in the proof of [76, (3.3)], this thesis finds that, for any $x \in (A^w Q)^\complement$,

$$S(a)(x) \lesssim b^{Nvj_0} b^{-\frac{v\ell(Q)}{q}} ||a||_{L^q(Q)}.$$

From this, the size condition of a, and Lemma 3.1.8(iv), this thesis deduces that, for any $x \in (A^w Q)^\complement$,

$$\begin{aligned} S(a)(x) &\lesssim b^{Nvj_0} b^{-\frac{v\ell(Q)}{q}} ||\mathbf{1}_Q||_X^{-1} \left|B_{v\ell(Q)+u}\right|^{\frac{1}{q}} \\ &\le b^{(N-\beta)vj_0 + (\frac{1}{q}-\beta)u} ||\mathbf{1}_Q||_X^{-1} \frac{|Q|^\beta}{b^{[\ell(Q)-j_0]v\beta}} \\ &\lesssim ||\mathbf{1}_Q||_X^{-1} \left[\frac{|Q|}{\rho(x - x_Q)}\right]^\beta \le ||\mathbf{1}_Q||_X^{-1} \left[\mathcal{M}(\mathbf{1}_Q)(x)\right]^\beta . \end{aligned}$$

Using this, this thesis obtains, for any $x \in \mathbb{R}^n$,

s4e5 (3.1.11)

$$\begin{aligned} S(f)(x) &\le \sum_{i\in\mathbb{N}} |\lambda_i| S(a_i)(x) \mathbf{1}_{A^w B^{(i)}}(x) + \sum_{i\in\mathbb{N}} |\lambda_i| S(a_i)(x) \mathbf{1}_{(A^w B^{(i)})^\complement}(x) \\ &\lesssim \left\{ \sum_{i\in\mathbb{N}} [|\lambda_i| S(a_i)(x) \mathbf{1}_{A^w B^{(i)}}(x)]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \\ &\quad + \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\mathbf{1}_{B^{(i)}}\|_X} \left[\mathcal{M}\left(\mathbf{1}_{B^{(i)}}\right)(x)\right]^\beta . \end{aligned}$$

By (3.1.11), Assumptions 1.2.10 and 1.2.12, and an argument similar to that used in the proof of [97, Theorem 4.3], this thesis further conclude that

$$\|S(f)\|_X \lesssim \|f\|_{H_X^A(\mathbb{R}^n)},$$

which completes the proof of the necessity of the present theorem.

Next, this thesis shows the sufficiency of the present theorem. Let ψ and ϕ be the same as in Lemma 3.1.2 with d in (2.2.1), f vanish weakly at infinity, and $\|S(f)\|_X < \infty$.

Then, from Theorem 3.1.7, this thesis infer that $S_\psi(f)\in X$. Thus, to show the sufficiency of the present theorem, this thesis needs to prove that $f\in H_X^A(\mathbb{R}^n)$ and

s4e6 (3.1.12)
$$\|f\|_{H_X^A(\mathbb{R}^n)}\lesssim \|S_\psi(f)\|_X\,.$$

To this end, for any $k\in\mathbb{Z}$, let $\Omega_k:=\{x\in\mathbb{R}^n: S_\psi(f)(x)>2^k\}$ and

$$\mathcal{Q}_k:=\left\{Q\in\mathcal{Q}:\ |Q\cap\Omega_k|>\frac{|Q|}{2}\ \text{and}\ |Q\cap\Omega_{k+1}|\le\frac{|Q|}{2}\right\}.$$

Clearly, for any $Q\in\mathcal{Q}$, there exists a unique $k\in\mathbb{Z}$ such that $Q\in\mathcal{Q}_k$. Let $\{Q_i^k\}_i$ be the set of all maximal dyadic cubes in $\mathcal{Q}_k$, that is, there exists no $Q\in\mathcal{Q}_k$ such that $Q_i^k\subsetneqq Q$ for any i. Note that $\{Q_i^k\}_i$ can be divided into two cases, finite set and countable set, so this thesis omit its index set here can discuss them one by one below.

For any $Q\in\mathcal{Q}$, let

s4e7 (3.1.13)
$$\begin{aligned}\widehat{Q}:=\big\{(y,t)\in\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,\infty):\\ y\in Q,\, b^{v\ell(Q)+u+\tau}\le t<b^{v[\ell(Q)-1]+u+\tau}\big\}.\end{aligned}$$

Obviously, $\{\widehat{Q}\}_{Q\in\mathcal{Q}}$ are mutually disjoint and

s4e8 (3.1.14)
$$\mathbb{R}^{n+1}_+=\bigcup_{k\in\mathbb{Z}}\bigcup_i B_{k,i},$$

where, for any $k\in\mathbb{Z}$ and i, $B_{k,i}:=\bigcup_{Q\subset Q_i^k,Q\in\mathcal{Q}_k}\widehat{Q}$. Then, by Lemma 3.1.8(ii) and (3.1.13), this thesis easily find that $\{B_{k,i}\}_{k\in\mathbb{Z},i}$ are also mutually disjoint.

On the other hand, ψ has the vanishing moments up to order d. From Lemma 2.2.1, the properties of tempered distributions (see, for instance, [40, Theorem 2.3.20]), and (3.1.14), this thesis deduces that, for any $f\in\mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity and satisfying $\|S(f)\|_X<\infty$ and for any $x\in\mathbb{R}^n$, this thesis has

s4e9 (3.1.15)
$$\begin{aligned}f(x)&=\sum_{k\in\mathbb{Z}}f*\psi_k*\phi_k(x)\\ &=\int_{\mathbb{R}^{n+1}_+}(f*\psi_t)(y)\phi_t(x-y)\,dy\,dm(t)\end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where $m(t)$ denotes the counting measure on $\mathbb{R}$, that is, for any set $E\subset\mathbb{R}$, $m(E)$ is the number of integers contained in E if E has only finitely many elements, or else $m(E):=\infty$. For any $k\in\mathbb{Z}$, i, and $x\in\mathbb{R}^n$, let

$$h_i^k(x):=\int_{B_{k,i}}(f*\psi_t)(y)\phi_t(x-y)\,dy\,dm(t).$$

Next, this thesis proves the sufficiency of the present theorem in three steps.

Step (1) The target of this step is to show that

s4e10 (3.1.16)
$$\sum_{k\in\mathbb{Z}}\sum_{i} h_i^k \text{ converges in } \mathcal{S}'(\mathbb{R}^n).$$

To this end, following the proofs of assertions (i) and (ii) in the proof of the sufficiency of [69, Theorem 3.4(i)] with some slight modifications, this thesis concludes that, for any given $q\in(\max\{p_0,1\},\infty)$,

(i) for any $k\in\mathbb{Z}$, i, and $x\in\mathbb{R}^n$,

$$h_i^k(x)=\sum_{Q\subset Q_i^k, Q\in\mathcal{Q}_k}\int_{\widehat{Q}}(f*\psi_t)(y)\phi_k(x-y)\,dy\,dm(t)$$

holds true in $L^q(\mathbb{R}^n)$ and hence also in $\mathcal{S}'(\mathbb{R}^n)$;

(ii) for any $k\in\mathbb{Z}$ and i, $h_i^k=\lambda_i^k a_i^k$ is a multiple of an anisotropic (X,q,d)-atom, where, for any $k\in\mathbb{Z}$ and i, $\lambda_i^k\sim 2^k||\mathbf{1}_{B_i^k}||_X$ with the positive equivalence constants independent of both k and i, and a_i^k is an anisotropic (X,q,d)-atom satisfying, for any $q\in(\max\{p_0,1\},\infty)$, $k\in\mathbb{Z}$, i, and $\gamma\in\mathbb{Z}_+^n$,

$$\text{supp } a_i^k\subset B_i^k:=x_{Q_i^k}+B_{v[\ell(Q_i^k)-1]+u+3\tau},$$

$$\|a_i^k\|_{L^q(\mathbb{R}^n)}\le\|\mathbf{1}_{B_i^k}\|_X^{-1}|B_i^k|^{\frac{1}{r}}, \text{ and } \int_{\mathbb{R}^n}a_i^k(x)x^\gamma\,dx=0.$$

To show (3.1.16), this thesis next consider two cases: $i\in\mathbb{N}$ and $i\in\{1,\dots,I\}$ with some $I\in\mathbb{N}$.

Case 1) $i\in\mathbb{N}$. In this case, to prove (3.1.16), by Lemma 3.1.12, it suffices to show that

s4e11 (3.1.17)
$$\lim_{l\to\infty}\left\|\sum_{l\le|k|\le m}\sum_{l\le i\le m}\lambda_i^k a_i^k\right\|_{H_X^A(\mathbb{R}^n)}=0.$$

Indeed, for any $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, by the estimate that $\left|Q_i^k\cap\Omega_k\right|\ge\frac{|Q_i^k|}{2}$, this thesis finds that, for any $x\in\mathbb{R}^n$,

$$\mathcal{M}\left(\mathbf{1}_{Q_i^k\cap\Omega_k}\right)(x)\gtrsim\fint_{Q_i^k}\mathbf{1}_{Q_i^k\cap\Omega_k}(y)\,dy=\frac{|Q_i^k\cap\Omega_k|}{|Q_i^k|}\ge\frac{1}{2}.$$

This, together with Assumption 1.2.10, further implies that, for any $l,m\in\mathbb{N}$,

s4e12 (3.1.18)
$$\left\|\sum_{l\le|k|\le m}\sum_{l\le i\le m}\left(2^k\mathbf{1}_{B_i^k}\right)^{\theta_0}\right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}}=\left\|\left[\sum_{l\le|k|\le m}\sum_{l\le i\le m}2^{k\theta_0}\left(\mathbf{1}_{B_i^k}\right)^2\right]^{\frac{1}{2}}\right\|_{X^{\frac{2}{\theta_0}}}^{\frac{2}{\theta_0}}$$

$$\lesssim \left\| \left\{ \sum_{l\le|k|\le m} \sum_{l\le i\le m} 2^{k\theta_0} \left[\mathcal{M}\left(\mathbf{1}_{Q_i^k\cap\Omega_k}\right)\right]^2 \right\}^{\frac{1}{2}} \right\|_{X^{\frac{2}{\theta_0}}}^{\frac{2}{\theta_0}}$$

$$\lesssim \left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{Q_i^k\cap\Omega_k}\right)^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}}.$$

In addition, from the fact that, for any $l, m \in \mathbb{N}$, $\sum_{l\le|k|\le m} \sum_{l\le i\le m} \lambda_i^k a_i^k \in H_X^A(\mathbb{R}^n)$, Lemma 2.2.3(i), and Definition 1.2.6(i), this thesis deduces that

s4e13 (3.1.19)
$$\left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \lambda_i^k a_i^k \right\|_{H_X^A(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left[\frac{\lambda_i^k \mathbf{1}_{B_i^k}}{\|\mathbf{1}_{B_i^k}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X$$

$$\sim \left\| \left[\sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{B_i^k}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X$$

$$= \left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{B_i^k}\right)^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}}.$$

On the other hand, it follows from Definition 1.2.4 that, for any $l, m \in \mathbb{N}$,

$$\left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X^{\theta_0} = \left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k\setminus\Omega_{k+1}} + 2^k \mathbf{1}_{\Omega_{k+1}}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X^{\theta_0}$$

$$\lesssim \left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X^{\theta_0}$$

$$+ \left(\frac{1}{2}\right)^{\theta_0} \left\| \left[\sum_{l\le|k|\le m} \left(2^{k+1} \mathbf{1}_{\Omega_{k+1}}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X^{\theta_0}.$$

Therefore, as $l \to \infty$, this thesis has

s4e14 (3.1.20)
$$\left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X \sim \left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X.$$

This, combined with (3.1.18) and (3.1.19), further implies that, as $l \to \infty$,

$$\left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \lambda_i^k a_i^k \right\|_{H_X^A(\mathbb{R}^n)}$$

$$\lesssim \left\| \sum_{l\le|k|\le m} \sum_{l\le i\le m} \left(2^k \mathbf{1}_{Q_i^k\cap\Omega_k}\right)^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} \lesssim \left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X$$

$$\sim \left\| \left[\sum_{l\le|k|\le m} \left(2^k \mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X$$

$$\le \left\| S_\psi(f) \left(\sum_{l\le|k|\le m} \mathbf{1}_{\Omega_k\setminus\Omega_{k+1}} \right)^{\frac{1}{\theta_0}} \right\|_X \to 0.$$

Thus, (3.1.17) holds true and so (3.1.16) does in Case 1).

Case 2) $i \in \{1,\dots,I\}$ with some $I\in\mathbb{N}$. In this case, to show (3.1.16), by Lemma 3.1.12, it suffices to prove that

s4e15 (3.1.21)
$$\lim_{l\to\infty} \left\| \sum_{l\le|k|\le m} \sum_{i=1}^{I} \lambda_i^k a_i^k \right\|_{H_X^A(\mathbb{R}^n)} = 0.$$

Indeed, by a proof similar to that of (3.1.17), it is easy to show that (3.1.21) also holds true. This finishes the proof of (3.1.16) in Case 2) and hence (3.1.16).

Step (2) In this step, this thesis proves that

s4e16 (3.1.22)
$$f = \sum_{k\in\mathbb{Z}} \sum_i \lambda_i^k a_i^k \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

To this end, for any $x\in\mathbb{R}^n$, let

$$\widetilde{f}(x) := \sum_{k\in\mathbb{Z}} \sum_i h_i^k(x) = \sum_{k\in\mathbb{Z}} \sum_i \int_{B_{k,i}} (f*\psi_t)(y)\phi_k(x-y)\,dy\,dm(t)$$

in $\mathcal{S}'(\mathbb{R}^n)$, where, for any $k\in\mathbb{Z}$ and $i, B_{k,i}$ is the same as in (3.1.14). Then, to show (3.1.22), it suffices to prove that

s4e16plus (3.1.23)
$$f = \widetilde{f} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

For this purpose, by the above assertion (i) and (3.1.13), this thesis finds that, for any given $k, i\in\mathbb{Z}$, $q\in(\max\{p_0,1\},\infty)$, and $x\in\mathbb{R}^n$,

s4e17 (3.1.24)
$$\begin{aligned} h_i^k(x) &= \lim_{N\to\infty} \int_0^\infty \int_{\mathbb{R}^n} (f*\psi_t)(y)\phi_k(x-y) \\ &\quad \times \mathbf{1}_{\bigcup_{Q\subset Q_i^k, Q\in\mathcal{Q}_k, |\ell(Q)|\le N} \widehat{Q}}(y,t)\,dy\,dm(t) \\ &= \lim_{N\to\infty} \int_{\gamma(N)}^{\eta(N)} \int_{\mathbb{R}^n} (f*\psi_t)(y)\phi_k(x-y)\mathbf{1}_{B_{k,i}}(y,t)\,dy\,dm(t) \end{aligned}$$

holds true in $L^q(\mathbb{R}^n)$ and also in $\mathcal{S}'(\mathbb{R}^n)$, where, for any $N \in \mathbb{N}$, $\gamma(N) := b^{vN+u+1}$ and $\eta(N) := b^{-v(N+1)+u+1}$. For the convenience of symbols, this thesis rewrite $\widetilde{f}$ as, for any $x \in \mathbb{R}^n$,

$$\widetilde{f}(x) = \sum_{\ell\in\mathbb{N}} \int_{R^{(\ell)}} (f * \psi_t)(y)\phi_t(x-y)\,dy\,dm(t),$$

where $\{R^{(\ell)}\}_{\ell\in\mathbb{N}}$ is an arbitrary permutation of $\{B_{k,i}\}_{k\in\mathbb{Z},i}$. For any $L \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$\widetilde{f}_L(x) := f(x) - \sum_{\ell=1}^{L} \int_{R^{(\ell)}} (f * \psi_t)(y)\phi_t(x-y)\,dy\,dm(t).$$

Then, from (3.1.14), (3.1.15), and (3.1.24), it follows that, for any $L \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

s4e18 (3.1.25)
$$\begin{aligned}\widetilde{f}_L(x) &= \lim_{N\to\infty} \int_{\gamma(N)}^{\eta(N)} \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=1}^{\infty} R^{(\ell)}}(y,t)\,dy\,dm(t)\\ &\quad - \lim_{N\to\infty} \int_{\gamma(N)}^{\eta(N)} \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=1}^{L} R^{(\ell)}}(y,t)\,dy\,dm(t)\\ &= \lim_{N\to\infty} \int_{\gamma(N)}^{\eta(N)} \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=L+1}^{\infty} R^{(\ell)}}(y,t)\,dy\,dm(t)\end{aligned}$$

holds true in $\mathcal{S}'(\mathbb{R}^n)$.

Note that $H_X^A(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$ (Lemma 3.1.12). Thus, to prove (3.1.23), this thesis only need to show that

s4e19 (3.1.26)
$$\left\|\widetilde{f}_L\right\|_{H_X^A(\mathbb{R}^n)} \to 0 \quad \text{as} \quad L \to \infty.$$

To do this, this thesis borrow some ideas from the proof of the atomic characterization of $H_X^A(\mathbb{R}^n)$ (see the proof of [97, Theorem 4.3]). Indeed, for any $\varepsilon \in (0,1), L \in \mathbb{N}$, and $x \in \mathbb{R}^n$, let

$$\widetilde{f}_L^{(\varepsilon)}(x) := \int_{\varepsilon}^{\alpha/\varepsilon} \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_t(x-y)\mathbf{1}_{\cup_{\ell=L+1}^{\infty} R^{(\ell)}}(y,t)\,dy\,dm(t),$$

where $\alpha := b^{-v+2(u+1)}$. Then, by the Lebesgue dominated convergence theorem, this thesis finds that, for any $\varepsilon \in (0,1), L \in \mathbb{N}$, and $x \in \mathbb{R}^n$,

$$\begin{aligned}\widetilde{f}_L^{(\varepsilon)}(x) &= \sum_{\ell=L+1}^{\infty} \int_{\varepsilon}^{\alpha/\varepsilon} \int_{\mathbb{R}^n} (f * \psi_t)(y)\phi_t(x-y)\mathbf{1}_{R^{(\ell)}}(y,t)\,dy\,dm(t)\\ &=: \sum_{\ell=L+1}^{\infty} h_\ell^{(\varepsilon)}(x)\end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, by some arguments similar to those used in the proofs of assertions (i) and (ii) in the proof of the sufficiency of [69, Theorem 3.4(i)] with some slight modifications, this thesis concludes that, for any $\varepsilon \in (0,1), q \in (\max\{p_0, 1\}, \infty), L \in \mathbb{N}$, and $\ell \in \mathbb{N} \cap [L+1, \infty), h_\ell^{(\varepsilon)}$ is a multiple of an anisotropic (X, q, d)-atom, that is, there exists a sequence $\{\lambda_\ell\}_{\ell \in \mathbb{N} \cap (L+1,\infty)} \subset [0,\infty)$ and a sequence $\{a_\ell^{(\varepsilon)}\}_{\ell \in \mathbb{N} \cap (L+1,\infty)}$ of anisotropic (X, q, d)-atoms supported, respectively, in $\{B^{(\ell)}\}_{\ell \in \mathbb{N} \cap (L+1,\infty)} \subset \mathcal{B}$ such that, for any $\ell \in \mathbb{N} \cap [L+1, \infty), h_\ell^{(\varepsilon)} = \lambda_\ell a_\ell^{(\varepsilon)}$, where, for any $\ell \in \mathbb{N} \cap [L+1, \infty), \lambda_\ell$ and $B^{(\ell)}$ are independent of ε. Therefore, for any $\varepsilon \in (0,1), L \in \mathbb{N}$, and $x \in \mathbb{R}^n$,

s4e20
$$\widetilde{f}_L^{(\varepsilon)}(x) = \sum_{\ell=L+1}^{\infty} \lambda_\ell a_\ell^{(\varepsilon)}(x) \text{ in } \mathcal{S}'(\mathbb{R}^n) \tag{3.1.27}$$

and

s4e21
$$\left\| \left\{ \sum_{\ell=L+1}^{\infty} \left[\frac{\lambda_\ell \mathbf{1}_{B^{(\ell)}}}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X < \infty. \tag{3.1.28}$$

On the other hand, for any given

$$N_0 \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{\theta_0} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right\rfloor + 2, \infty \right),$$

let $M_{N_0}^0$ denote the anisotropic radial grand maximal function in Definition 3.1.1 with N replaced by N_0. Then, by the just proved conclusion that, for any $\varepsilon \in (0,1)$ and $L \in \mathbb{N}, \{a_\ell^{(\varepsilon)}\}_{\ell \in \mathbb{N} \cap (L+1,\infty)}$ is a sequence of anisotropic (X, q, d)-atoms and [97, Lemma 4.7], this thesis finds that, for any $\ell \in \mathbb{N} \cap [L+1, \infty)$ and $x \in \mathbb{R}^n$,

s4e22
$$M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right)(x) \lesssim M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right)(x) \mathbf{1}_{A^\tau B^{(\ell)}}(x) + \frac{1}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \left[\mathcal{M}\left(\mathbf{1}_{B^{(\ell)}} \right)(x) \right]^\beta, \tag{3.1.29}$$

where $\beta := \left(\frac{\ln b}{\ln \lambda_-} + d + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{\theta_0}$. Moreover, since $q > 1$, then, from the boundedness of $\mathcal{M}$ on $L^q(\mathbb{R}^n)$ (see [75, Lemma 3.3(ii)]), this thesis deduces that, for any $\varepsilon \in (0,1), L \in \mathbb{N}$, and $\ell \in \mathbb{N} \cap [L+1, \infty)$,

$$\left\| M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right) \mathbf{1}_{A^\tau B^{(\ell)}} \right\|_{L^q(\mathbb{R}^n)} \lesssim \left\| \mathcal{M}\left(a_\ell^{(\varepsilon)} \right) \mathbf{1}_{A^\tau B^{(\ell)}} \right\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|B^{(\ell)}|^{1/q}}{\|\mathbf{1}_{B^{(\ell)}}\|_X},$$

which, combined with Lemma 3.1.14, further implies that

s4e23
$$\begin{aligned} &\left\| \liminf_{\varepsilon \to 0^+} \left\{ \sum_{\ell=L+1}^{\infty} \left[\lambda_\ell M_{N_0}^0 \left(a_\ell^{(\varepsilon)} \right) \mathbf{1}_{A^\tau B^{(\ell)}} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X \\ &\quad \lesssim \left\| \left\{ \sum_{\ell=L+1}^{\infty} \left[\frac{\lambda_\ell \mathbf{1}_{B^{(\ell)}}}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X. \end{aligned} \tag{3.1.30}$$

In addition, let $\varepsilon := \gamma(N)$ with $N \in \mathbb{N} \cap [\lfloor \frac{-u-1}{v} \rfloor + 1, \infty)$. Then, by (3.1.25), this thesis obtains, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} M_{N_0}^0\left(\widetilde{f}_L\right)(x) &= M_{N_0}^0\left(\lim_{N\to\infty} \widetilde{f}_L^{(\gamma(N))}\right)(x) \\ &= \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} \sup_{k\in\mathbb{Z}} \left| \lim_{N\to\infty} \widetilde{f}_L^{(\gamma(N))} * \varphi_k(x) \right| \\ &\le \liminf_{N\to\infty} \sup_{\varphi \in \mathcal{S}_N(\mathbb{R}^n)} \sup_{k\in\mathbb{Z}} \left| \widetilde{f}_L^{(\gamma(N))} * \varphi_k(x) \right| \\ &= \liminf_{N\to\infty} M_{N_0}^0\left(\widetilde{f}_L^{(\gamma(N))}\right). \end{aligned}$$

From this, [4, p. 12, Proposition 3.10], (3.1.27), and (3.1.29), it follows that, for any $L \in \mathbb{N}$,

$$\begin{aligned} \left\|\widetilde{f}_L\right\|_{H_X^A(\mathbb{R}^n)} &\le \left\| \liminf_{N\to\infty} M_{N_0}^0\left(\widetilde{f}_L^{(\gamma(N))}\right)\right\|_X \\ &\le \left\| \liminf_{N\to\infty} \sum_{\ell=L+1}^{\infty} \lambda_\ell M_{N_0}^0\left(a_\ell^{(\gamma(N))}\right)\right\|_X \\ &\lesssim \left\| \liminf_{N\to\infty} \sum_{\ell=L+1}^{\infty} \lambda_\ell M_{N_0}^0\left(a_\ell^{(\gamma(N))}\right)\mathbf{1}_{A^\tau B^{(\ell)}}\right\|_X \\ &\quad + \left\| \sum_{\ell=L+1}^{\infty} \frac{\lambda_\ell}{\|\mathbf{1}_{A^\tau B^{(\ell)}}\|_X} \left[\mathcal{M}\left(\mathbf{1}_{B^{(\ell)}}\right)\right]^\beta \right\|_X. \end{aligned}$$

This, together with (3.1.30), Lemma 2.1.8, Definition 1.2.4(ii), Assumption 1.2.10, and $\beta > \frac{1}{\theta_0}$, further implies that, for any $L \in \mathbb{N}$,

$$\begin{aligned} \left\|\widetilde{f}_L\right\|_{H_X^A(\mathbb{R}^n)} &\lesssim \left\| \liminf_{N\to\infty} \left\{ \sum_{\ell=L+1}^{\infty} \left[\lambda_\ell M_{N_0}^0\left(a_\ell^{(\gamma(N))}\right)\mathbf{1}_{A^\tau B^{(\ell)}}\right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X \\ &\quad + \left\| \left\{ \sum_{\ell=L+1}^{\infty} \frac{\lambda_\ell}{\|\mathbf{1}_{B^{(\ell)}}\|_X} \left[\mathcal{M}\left(\mathbf{1}_{B^{(\ell)}}\right)\right]^\beta \right\}^{\frac{1}{\beta}} \right\|_{X^\beta}^\beta \\ &\lesssim \left\| \left\{ \sum_{\ell=L+1}^{\infty} \left[\frac{\lambda_\ell \mathbf{1}_{B^{(\ell)}}}{\|\mathbf{1}_{B^{(\ell)}}\|_X}\right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X. \end{aligned}$$

By this and (3.1.28), this thesis concludes that (3.1.26) holds true, which completes the proof of (3.1.23) and hence (3.1.22).

Step (3) By (3.1.22), [97, Theorem 4.3], and some arguments similar to those used in the estimations of both (3.1.18) and (3.1.20), this thesis concludes that

$$\|f\|_{H_X^A(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_i \left[\frac{\lambda_i^k \mathbf{1}_{B_i^k}}{\|\mathbf{1}_{B_i^k}\|_X}\right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X = \left\| \left[\sum_{k\in\mathbb{Z}} \sum_i \left(2^k \mathbf{1}_{B_i^k}\right)^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X$$

$$
\begin{aligned}
&\lesssim \left\|\sum_{k\in\mathbb{Z}}\sum_{i}\left(2^k\mathbf{1}_{Q_i^k\cap\Omega_k}\right)^{\theta_0}\right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} \le \left\|\left[\sum_{k\in\mathbb{Z}}\left(2^k\mathbf{1}_{\Omega_k}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X\\
&\sim \left\|\left[\sum_{k\in\mathbb{Z}}\left(2^k\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right)^{\theta_0}\right]^{\frac{1}{\theta_0}}\right\|_X \le \left\|S_\psi(f)\left[\sum_{k\in\mathbb{Z}}\mathbf{1}_{\Omega_k\setminus\Omega_{k+1}}\right]^{\frac{1}{\theta_0}}\right\|_X\\
&= \|S_\psi(f)\|_X\,,
\end{aligned}
$$

which further implies that $f\in H_X^A(\mathbb{R}^n)$ and (3.1.12) holds true. This finishes the proof the sufficiency and hence Theorem 3.1.4. $\square$

Now, this thesis establishes the anisotropic Littlewood–Paley g-function characterization of $H_X^A(\mathbb{R}^n)$. Recall that, for any given dilation A, $\phi\in\mathcal{S}(\mathbb{R}^n)$, $t\in(0,\infty)$, and $j\in\mathbb{Z}$ and for any $f\in\mathcal{S}'(\mathbb{R}^n)$, the *anisotropic Peetre maximal function* $(\phi_j^*f)_t$ is defined by setting, for any $x\in\mathbb{R}^n$,

$$(\phi_j^*f)_t(x) := \operatorname*{ess\,sup}_{y\in\mathbb{R}^n}\frac{|(\phi_{-j}*f)(x+y)|}{[1+b^j\rho(y)]^t}$$

and the *g-function associated with* $(\phi_j^*f)_t$ is defined by setting, for any $x\in\mathbb{R}^n$,

$$g_{t,*}(f)(x) := \left\{\sum_{j\in\mathbb{Z}}\left[\left(\phi_j^*f\right)_t(x)\right]^2\right\}^{1/2}.$$

To prove Theorem 3.1.5, this thesis needs the following estimate which is just [73, Lemma 3.6] originated from [93, (2.66)].

s414 **Lemma 3.1.15.** *Let ϕ be a radial function the same as in Lemma 3.1.2. Then, for any given $N_0\in\mathbb{N}$ and $\gamma\in(0,\infty)$, there exists a positive constant $C_{(N_0,\gamma)}$, depending only on N_0 and γ, such that, for any $t\in(0,N_0)$, $l\in\mathbb{Z}$, $f\in\mathcal{S}'(\mathbb{R}^n)$, and $x\in\mathbb{R}^n$,*

$$\left[\left(\phi_l^*f\right)_t(x)\right]^\gamma \le C_{(N_0,\gamma)}\sum_{k=0}^{\infty}b^{-kN_0\gamma}b^{k+l}\int_{\mathbb{R}^n}\frac{|\phi_{-(k+l)}*f(y)|^\gamma}{[1+b^l\rho(x-y)]^{t\gamma}}\,dy.$$

This thesis now prove Theorem 3.1.5.

Proof of Theorem 3.1.5. First, let $f\in H_X^A(\mathbb{R}^n)$. Then, by Lemma 3.1.13, this thesis finds that f vanishes weakly at infinity. In addition, repeating the proof of the necessity of Theorem 3.1.4 with some slight modifications, this thesis easily find that $g(f)\in X$ and $||g(f)||_X\lesssim||f||_{H_X^A(\mathbb{R}^n)}$. Thus, to prove the present theorem, by Theorem 3.1.4, this thesis only need to show that, for any $f\in\mathcal{S}'(\mathbb{R}^n)$ satisfying that f vanishes weakly at infinity and $g(f)\in X$,

s4e25 (3.1.31) $$\|S(f)\|_X\lesssim\|g(f)\|_X$$

holds true. Notice that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity, any $t \in (0,\infty)$, and almost every $x \in \mathbb{R}^n$, $S(f)(x) \lesssim g_{t,*}(f)(x)$. Thus, to show (3.1.31), it suffices to prove that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity,

s4e26
$$\|g_{t,*}(f)\|_X \lesssim \|g(f)\|_X \tag{3.1.32}$$

holds true for some $t \in (1/r_+,\infty)$ with r_+ the same as in (3.1.4). Now, this thesis shows (3.1.32). To this end, assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function the same as in Lemma 3.1.2. Obviously, $t \in (1/r_+,\infty)$ implies that there exists a $\theta_0 \in (0,r_+)$ such that $t \in (1/\theta_0,\infty)$. Fix an $N_0 \in (1/\theta_0,\infty)$. By this, Lemma 3.1.15, and the Minkowski inequality, this thesis finds that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
g_{t,*}(f)(x) &= \left\{\sum_{k\in\mathbb{Z}} \left[(\phi_k^* f)_t(x)\right]^2\right\}^{\frac{1}{2}} \\
&\lesssim \left[\sum_{k\in\mathbb{Z}}\left\{\sum_{j\in\mathbb{Z}_+} b^{-jN_0r_+} b^{j+k}\int_{\mathbb{R}^n}\frac{|(\phi_{-(j+k)}*f)(y)|^{r_+}}{[1+b^k\rho(x-y)]^{tr_+}}\,dy\right\}^{\frac{2}{r_+}}\right]^{\frac{1}{2}} \\
&\le \left\{\sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)}\right. \\
&\quad \left.\times\left[\sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}}\left\{\int_{\mathbb{R}^n}\frac{|(\phi_{-(j+k)}*f)(y)|^{r_+}}{[1+b^k\rho(x-y)]^{tr_+}}\,dy\right\}^{\frac{2}{r_+}}\right]^{\frac{r_+}{2}}\right\}^{\frac{1}{r_+}},
\end{aligned}$$

which further implies that

$$\begin{aligned}
\|g_{t,*}(f)\|_X^{r_+\theta_0} &\lesssim \left\|\sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)}\right. \\
&\quad \left.\times\left[\sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}}\left\{\int_{\mathbb{R}^n}\frac{|(\phi_{-(j+k)}*f)(y)|^{r_+}}{[1+b^k\rho(\cdot-y)]^{tr_+}}\,dy\right\}^{\frac{2}{r_+}}\right]^{\frac{r_+}{2}}\right\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\le \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0}\left\|\left\{\sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}}\right.\right. \\
&\quad \times\left[\left(\int_{\{y\in\mathbb{R}^n:\,\rho(\cdot-y)<b^{-k}\}}+\sum_{i\in\mathbb{Z}_+} b^{-itr_+}\int_{\{y\in\mathbb{R}^n:\,b^{i-k-1}<\rho(\cdot-y)<b^{i-k}\}}\right)\right. \\
&\quad \left.\left.\left.\times\left|\left(\phi_{-(j+k)}*f\right)(y)\right|^{r_+}\,dy\right]^{\frac{2}{r_+}}\right\}^{\frac{r_+}{2}}\right\|_{X^{\frac{1}{r_+}}}^{\theta_0}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \Bigg\| \sum_{k\in\mathbb{Z}} b^{\frac{2k}{r_+}} \Bigg\{ \sum_{i\in\mathbb{N}} b^{-itr_+} \\
&\quad \times \left[\int_{\{y\in\mathbb{R}^n:\,\rho(\cdot-y)<b^{-k}\}} \left|\left(\phi_{-(j+k)}*f\right)(y)\right|^{r_+}\,dy\right]^{\frac{2}{r_+}}\Bigg\}^{\frac{r_+}{2}}\Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0}.
\end{aligned}
$$

Then, from the Minkowski inequality again and Assumption 1.2.10, this thesis further infer that

$$
\begin{aligned}
\|g_{t,*}(f)\|_X^{r_+\theta_0} &\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \Bigg\| \sum_{i\in\mathbb{N}} b^{-itr_+} \Bigg\{ \sum_{k\in\mathbb{Z}} b^{k} \\
&\quad \times \left[\int_{\{y\in\mathbb{R}^n:\,\rho(\cdot-y)<b^{-k}\}} \left|\left(\phi_{-(j+k)}*f\right)(y)\right|^{r_+}\,dy\right]^{\frac{2}{r_+}}\Bigg\}^{\frac{r_+}{2}}\Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\le \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \\
&\quad \times \Bigg\| \sum_{i\in\mathbb{N}} b^{(1-tr_+)i} \Bigg\{ \sum_{k\in\mathbb{Z}} \left[\mathcal{M}\left(\left|\phi_{-(j+k)}*f\right|^{r_+}\right)\right]^{\frac{2}{r_+}}\Bigg\}^{\frac{r_+}{2}}\Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r_+-1)\theta_0} \sum_{i\in\mathbb{N}} b^{(1-tr_+)i\theta_0} \\
&\quad \times \Bigg\| \Bigg\{ \sum_{k\in\mathbb{Z}} \left[\left|\phi_{-(j+k)}*f\right|^{r_+}\right]^{\frac{2}{r_+}}\Bigg\}^{\frac{r_+}{2}}\Bigg\|_{X^{\frac{1}{r_+}}}^{\theta_0} \\
&\sim \|g(f)\|_X^{r_+\theta_0}.
\end{aligned}
$$

This further implies that (3.1.32) holds true and hence finishes the proof of Theorem 3.1.5. □

3.24.x3 **Remark 3.1.16.** (i) If $A := 2\,I_{n\times n}$, then Theorems 3.1.4, 3.1.5, and 3.1.6 were obtained in [17, Theorems 4.9, 4.11, and 4.13] (see also [81, Theorem 3.21] and [95, Theorem 2.10]).

(ii) As was mentioned in Remark 2.2.8(ii), although $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type, Theorems 3.1.4, 3.1.5, and 3.1.6 can not be deduced from [98, Theorems 4.11, 5.1, and 5.3] and, actually, they can not cover each other.

c3s2 3.2 Fourier Transforms of $H^A_X(\mathbb{R}^n)$

In this section, this thesis aim to study the Fourier transform of f. Recall that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, its *Fourier transform*, denoted by $\mathscr{F}(\varphi)$ or $\widehat{\varphi}$, is defined by setting, for any $\xi \in \mathbb{R}^n$,

$$\mathscr{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi \imath x\cdot\xi}\, dx,$$

here and thereafter, $\imath := \sqrt{-1}$ and, for any $x := (x_1, \ldots, x_n), \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x \cdot \xi := \sum_{i=1}^n x_i \xi_i$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ is defined by setting, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{f}, \varphi\rangle := \langle f, \widehat{\varphi}\rangle$; also, for any $f \in \mathcal{S}(\mathbb{R}^n)$ [resp. $\mathcal{S}'(\mathbb{R}^n)$], $f^\vee$ denotes its *inverse Fourier transform* which is defined by setting, for any $\xi \in \mathbb{R}^n$, $f^\vee(\xi) := \widehat{f}(-\xi)$ [resp. for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle f^\vee, \varphi\rangle := \langle f, \varphi^\vee\rangle$].

Now, this thesis present the main result of this section as follows.

c3s2t1 **Theorem 3.2.1.** *Let A be a dilation and X a ball quasi-Banach function space satisfying both Assumption 1.2.10 with $p_- \in (0,\infty)$ and Assumption 1.2.12 with the same p_-, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\theta_0, \infty)$, where $\underline{p}$ is the same as in* (1.2.6)*. Further assume that there exists $q_0 \in [\theta_0, 1]$ such that:*

(i) *for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$,*

6.9.x1 (3.2.1)
$$\sum_{k=1}^\infty \|f_k\|_{X^{\frac{1}{q_0}}} \lesssim \left\|\sum_{k=1}^\infty f_k\right\|_{X^{\frac{1}{q_0}}},$$

where the implicit positive constant is independent of $\{f_k\}_{k=1}^\infty$;

(ii) *for any $B \in \mathcal{B}$ with $\mathcal{B}$ in* (1.2.3)*,*

one_B_X (3.2.2)
$$\|\mathbf{1}_B\|_X \gtrsim \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\},$$

where the implicit positive constant is independent of B.

Then, for any $f \in H^A_X(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that

6.10.x3 (3.2.3)
$$\widehat{f} = F \quad in \quad \mathcal{S}'(\mathbb{R}^n)$$

and there exists a positive constant C, depending only on A and X, such that, for any $x \in \mathbb{R}^n$,

c3s2e1 (3.2.4)
$$|F(x)| \le C\|f\|_{H^A_X(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1}, [\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\},$$

here and thereafter, ρ_* is defined as in Definition 1.2.3 with A replaced by its transposed matrix A^T.

c3s2re1 **Remark 3.2.2.** (i) If $A := 2\,I_{n\times n}$, then Theorem 3.2.1 was obtained in [46, Theorem 2.1].

(ii) For any given measurable set $E \subset \mathbb{R}^n$ and any given $p \in (0,\infty)$, the *Lebesgue space* $L^p(E)$ is defined by setting,

6.11.y1
$$L^p(E) := \left\{ f \text{ is measurable on } E :\ \|f\|_{L^p(E)} := \left[\int_E |f(x)|^p\,dx\right]^{1/p} < \infty \right\}. \tag{3.2.5}$$

Let A be a dilation, $p \in (0,1)$, and

$$N \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{p}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2, \infty\right).$$

Then, by [99, Remarks 2.7(i) and 4.21(i)], this thesis concludes that $L^p(\mathbb{R}^n)$ satisfies all assumptions of Definition 2.1.1 with $X := L^p(\mathbb{R}^n)$, $p_- \in (0,p]$, $\theta_0 \in (0,p_-)$, and $p_0 \in (p,\infty)$. Moreover, choose $q_0 \in (p,1]$. Then it follows from (3.2.5) that, for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$ and any $B \in \mathcal{B}$,

$$\sum_{k=1}^\infty \|f_k\|_{L^{\frac{p}{q_0}}(\mathbb{R}^n)} \le \left\|\sum_{k=1}^\infty f_k\right\|_{L^{\frac{p}{q_0}}(\mathbb{R}^n)}$$

and

$$\|\mathbf{1}_B\|_{L^p(\mathbb{R}^n)} = |B|^{\frac{1}{p}} > \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\}.$$

Thus, $L^p(\mathbb{R}^n)$ satisfies all the assumptions of Theorem 3.2.1 with $X := L^p(\mathbb{R}^n)$. In this case, Theorem 3.2.1 was obtained in [9, Theorem 1].

(iii) As mentioned in [46, Remark 2.1(ii)], (3.2.4) implies that the function $f \in H_X^A(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has a vanishing moment. This illustrates the necessity of the vanishing moment of atoms in some sense.

To prove Theorem 3.2.1, this thesis needs more preparations. Let A be a dilation. Recall that the dilation operator D_A is defined by setting, for any $f \in \mathscr{M}(\mathbb{R}^n)$,

$$D_A(f)(\cdot) := f(A\cdot).$$

Then, by an elementary calculation (see also [9, (3.1)]), this thesis finds that, for any $k \in \mathbb{Z}$, $f \in L^1(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

6.10.x1
$$\widehat{f}(x) = b^k \left(D_{A^*}^k\left(\mathscr{F}\left(D_A^k f\right)\right)\right)(x). \tag{3.2.6}$$

Next, this thesis recall the definition of anisotropic atomic Hardy spaces which were first introduced in [97, Definition 4.2].

atom **Definition 3.2.3.** Let A, X, θ_0, and p_0 be the same as in Definition 2.1.1. Further assume that $q \in (\max\{p_0, 1\}, \infty]$ and

$$d \in \left[\left\lfloor \left(\frac{1}{\theta_0} - 1\right) \frac{\ln b}{\ln(\lambda_-)} \right\rfloor, \infty\right) \cap \mathbb{Z}_+. \tag{3.2.7}$$

The *anisotropic atomic Hardy space* $H^{A,q,d}_{X,\mathrm{atom}}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_j\}_{j\in\mathbb{N}}$ of (X, q, d)-atoms supported, respectively, in $\{B^j\}_{j\in\mathbb{N}} \subset \mathcal{B}$ such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j$$

in $\mathcal{S}'(\mathbb{R}^n)$ and that

$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j| \mathbf{1}_{B^j}}{\|\mathbf{1}_{B^j}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X < \infty.$$

Moreover, for any $f \in H^{A,q,d}_{X,\mathrm{atom}}(\mathbb{R}^n)$, let

$$\|f\|_{H^{A,q,d}_{X,\mathrm{atom}}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j| \mathbf{1}_{B^j}}{\|\mathbf{1}_{B^j}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X,$$

where the infimum is taken over all decompositions of f as above.

The following atomic characterization of $H^A_X(\mathbb{R}^n)$, which was established in [97, Theorem 4.3], is needed in the proof of Theorem 3.2.1.

c3s2l0 **Lemma 3.2.4.** *Let A, X, q, and d be the same as in Definition 3.2.3. Then $H^A_X(\mathbb{R}^n) = H^{A,q,d}_{X,\mathrm{atom}}(\mathbb{R}^n)$ with equivalent quasi-norms.*

By an argument similar to that used in proof of [9, Lemma 4], this thesis immediately obtain the following conclusion.

c3s2l1 **Lemma 3.2.5.** *Let A, X, q, and d be the same as in Definition 3.2.3. Assume that a is an anisotropic (X, q, d)-atom supported in $x_0 + B_{i_0}$ with some $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathbb{Z}$. Then there exists a positive constant C such that, for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le d$ and for any $x \in \mathbb{R}^n$,*

5.30.x2

$$\left|\partial^\alpha \left(\mathscr{F}\left(D^{i_0}_A a\right)\right)(x)\right| \le C \left\|\mathbf{1}_{B_{i_0}}\right\|_X^{-1} \min\left\{1, |x|^{d-|\alpha|+1}\right\}, \tag{3.2.8}$$

where C is also independent of a.

Proof. Without loss of generality, this thesis may assume that a is supported in B_{i_0}. Thus, $\operatorname{supp}(D_A^{i_0}a) \subset B_0$. On the one hand, by [31, (1.20)], Definition 3.2.3(i)$_3$, the Taylor remainder theorem, the Hölder inequality, and Definition 3.2.3(i)$_2$, this thesis concludes that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le d$ and for any $x \in \mathbb{R}^n$,

5.30.x1 (3.2.9)
$$\begin{aligned}\left|\partial^\alpha\left(\mathscr{F}\left(D_A^{i_0}a\right)\right)(x)\right| &= \left|\int_{B_0}(-2\pi\imath\xi)^\alpha\left(D_A^{i_0}a\right)(\xi)e^{-2\pi\imath x\cdot\xi}\,d\xi\right|\\ &= \left|\int_{B_0}(-2\pi\imath\xi)^\alpha\left(D_A^{i_0}a\right)(\xi)\left[e^{-2\pi\imath x\cdot\xi}-T(\xi)\right]\,d\xi\right|\\ &\lesssim \int_{B_0}|\xi|^{|\alpha|}\left|a\left(A^{i_0}\xi\right)\right||x|^{d-|\alpha|+1}|\xi|^{d-|\alpha|+1}\,d\xi\\ &\lesssim |x|^{d-|\alpha|+1}b^{-i_0}\int_{B_{i_0}}|a(\xi)|\,d\xi \le |x|^{d-|\alpha|+1}\left\|\mathbf{1}_{B_{i_0}}\right\|_X^{-1},\end{aligned}$$

where $T(\xi)$ is the $(d-|\alpha|)$th-order Taylor polynomial of the function $\xi \to e^{-2\pi\imath x\cdot\xi}$ at the origin. On the other hand, from [31, (1.20)], the Hölder inequality, and Definition 3.2.3(i)$_2$, this thesis deduces that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le d$ and for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\left|\partial^\alpha\left(\mathscr{F}\left(D_A^{i_0}a\right)\right)(x)\right| &= \left|\int_{B_0}(-2\pi\imath\xi)^\alpha\left(D_A^{i_0}a\right)(\xi)e^{-2\pi\imath x\cdot\xi}\,d\xi\right|\\ &\lesssim \int_{B_0}|\xi|^{|\alpha|}\left|a\left(A^{i_0}\xi\right)\right|\,d\xi \lesssim b^{-i_0}\int_{B_{i_0}}|a(\xi)|\,d\xi\\ &\le \left\|\mathbf{1}_{B_{i_0}}\right\|_X^{-1},\end{aligned}$$

which, combined with (3.2.9), further implies (3.2.8) and hence completes the proof of Lemma 3.2.5. □

Applying Lemma 3.2.5, this thesis obtains the following uniform estimate on anisotropic (X, q, d)-atoms, which plays a key role in the proof of Theorem 3.2.1.

c3s2l2 **Lemma 3.2.6.** *Let A, X, q, d, and θ_0 be the same as in Definition 3.2.3. Further assume that X satisfies* (3.2.2) *with $q_0 \in [\theta_0, 1]$. Then there exists a positive constant C such that, for any anisotropic (X, q, d)-atom a and for any $x \in \mathbb{R}^n$,*

c3s2e2 (3.2.10)
$$|\widehat{a}(x)| \le C\max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1},\,[\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\},$$

where ρ_* is the same as in Theorem 3.2.1.

The proof of Lemma 3.2.6 needs the following inequalities which are just [4, p. 11, Lemma 3.2].

c3s2l3 **Lemma 3.2.7.** *Let A be a dilation. Then there exists a positive constant C such that, for any $x \in \mathbb{R}^n$,*

$$\frac{1}{C}[\rho(x)]^{\ln(\lambda_-)/\ln b} \le |x| \le C[\rho(x)]^{\ln(\lambda_+)/\ln b} \textit{ when } \rho(x) \in (1, \infty)$$

and

$$\frac{1}{C}[\rho(x)]^{\ln(\lambda_+)/\ln b} \le |x| \le C[\rho(x)]^{\ln(\lambda_-)/\ln b} \text{ when } \rho(x) \in [0,1].$$

Now, this thesis give the proof of Lemma 3.2.6.

Proof of Lemma 3.2.6. Let a be an anisotropic (X, q, d)-atom supported in $x_0 + B_{i_0}$ with some $x_0 \in \mathbb{R}^n$ and $i_0 \in \mathbb{Z}$. Without loss of generality, this thesis may assume $x_0 = \mathbf{0}$. By (3.2.6), Lemma 3.2.5 with $\alpha = (\overbrace{0, \dots, 0}^{n \text{ times}})$, and (3.2.2), this thesis concludes that, for any $x \in \mathbb{R}^n$,

c3s2e3 (3.2.11)
$$\begin{aligned} |\widehat{a}(x)| &= \left| b^{i_0} \left(D^{i_0}_{A^*} \left(\mathscr{F} \left(D^{i_0}_{A} a \right) \right) \right) (x) \right| = \left| b^{i_0} \mathscr{F} \left(D^{i_0}_{A} a \right) \left((A^*)^{i_0} x \right) \right| \\ &\lesssim b^{i_0} \left\| \mathbf{1}_{B_{i_0}} \right\|_X^{-1} \min\left\{ 1, \left|(A^*)^{i_0} x\right|^{d+1} \right\} \\ &\lesssim b^{i_0} \max\left\{ b^{-\frac{i_0}{q_0}}, b^{-\frac{i_0}{\theta_0}} \right\} \min\left\{ 1, \left|(A^*)^{i_0} x\right|^{d+1} \right\}. \end{aligned} \tag{3.2.11}$$

Next, this thesis proves (3.2.10) by considering the following two cases on $\rho_*(x)$.

Case 1) $\rho_*(x) \le b^{-i_0}$. In this case, note that

6.10.x2 (3.2.12)
$$\rho_* \left((A^*)^{i_0} x \right) = b^{i_0} \rho_*(x) \le 1. \tag{3.2.12}$$

Moreover, by (2.2.1), this thesis finds that

$$1 - \frac{1}{q_0} + (d+1)\frac{\ln(\lambda_-)}{\ln b} \ge 1 - \frac{1}{\theta_0} + (d+1)\frac{\ln(\lambda_-)}{\ln b} > 0.$$

From this, (3.2.11), (3.2.12), and Lemma 3.2.7, this thesis infers that, for any $x \in \mathbb{R}^n$ satisfying $\rho_*(x) \le b^{-i_0}$,

c3s2e4 (3.2.13)
$$\begin{aligned} |\widehat{a}(x)| &\lesssim b^{i_0} \max\left\{ b^{-\frac{i_0}{q_0}}, b^{-\frac{i_0}{\theta_0}} \right\} \left[\rho_* \left((A^*)^{i_0} x \right) \right]^{(d+1)\frac{\ln(\lambda_-)}{\ln b}} \\ &= \max\left\{ b^{i_0[1-\frac{1}{q_0}+(d+1)\frac{\ln(\lambda_-)}{\ln b}]}, b^{i_0[1-\frac{1}{\theta_0}+(d+1)\frac{\ln(\lambda_-)}{\ln b}]} \right\} [\rho_*(x)]^{(d+1)\frac{\ln(\lambda_-)}{\ln b}} \\ &= \max\left\{ [\rho_*(x)]^{\frac{1}{q_0}-1}, [\rho_*(x)]^{\frac{1}{\theta_0}-1} \right\}. \end{aligned} \tag{3.2.13}$$

This shows (3.2.10) in Case 1).

Case 2) $\rho_*(x) > b^{-i_0}$. In this case, note that $\rho_*((A^*)^{i_0} x) = b^{i_0} \rho_*(x) > 1$. Using this, (3.2.11), Lemma 3.2.7, and the fact that $\frac{1}{\theta_0} - 1 \ge \frac{1}{q_0} - 1 \ge 0$, this thesis concludes that, for any $x \in \mathbb{R}^n$ satisfying $\rho_*(x) > b^{-i_0}$,

$$\begin{aligned} |\widehat{a}(x)| &\lesssim b^{i_0} \max\left\{ b^{-\frac{i_0}{q_0}}, b^{-\frac{i_0}{\theta_0}} \right\} = \max\left\{ b^{-i_0(\frac{1}{q_0}-1)}, b^{-i_0(\frac{1}{\theta_0}-1)} \right\} \\ &\le \max\left\{ [\rho_*(x)]^{\frac{1}{q_0}-1}, [\rho_*(x)]^{\frac{1}{\theta_0}-1} \right\}, \end{aligned}$$

which, combined with (3.2.13), then completes the proof of (3.2.10) and hence Lemma 3.2.6. □

The following conclusion is also used in the proof of Theorem 3.2.1.

c3s214 **Lemma 3.2.8.** *Let A, X, and θ_0 be the same as in Definition 3.2.3. Further assume that X satisfies* (3.2.1) *with $q_0 \in [\theta_0, 1]$. Then there exists a positive constant C such that, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$,*

$$\sum_{i\in\mathbb{N}} |\lambda_i| \le C \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X .$$

Proof. Indeed, by Lemma 2.1.8, Definition 1.2.6, (3.2.1), and Definition 1.2.4(ii), this thesis finds that, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$,

$$\begin{aligned}
\sum_{i=1}^{\infty} |\lambda_i| &\le \left(\sum_{i=1}^{\infty} |\lambda_i|^{q_0} \right)^{\frac{1}{q_0}} = \left\{ \sum_{i=1}^{\infty} \left\| \frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right\|_X^{q_0} \right\}^{\frac{1}{q_0}} \\
&= \left\{ \sum_{i=1}^{\infty} \left\| \frac{|\lambda_i|^{q_0} \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X^{q_0}} \right\|_{X^{\frac{1}{q_0}}} \right\}^{\frac{1}{q_0}} \lesssim \left\| \sum_{i=1}^{\infty} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{q_0} \right\|_{X^{\frac{1}{q_0}}}^{\frac{1}{q_0}} \\
&= \left\| \left\{ \sum_{i=1}^{\infty} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{q_0} \right\}^{\frac{1}{q_0}} \right\|_X \le \left\| \left\{ \sum_{i=1}^{\infty} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X .
\end{aligned}$$

This finishes the proof of Lemma 3.2.8. $\square$

Next, this thesis shows Theorem 3.2.1.

Proof of Theorem 3.2.1. Let q and d be the same as in Definition 3.2.3. Without loss of generality, this thesis may assume that $\|f\|_{H_X^A(\mathbb{R}^n)} > 0$. Then, by Lemma 3.2.4 and Definition 3.2.3(ii), this thesis finds that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of anisotropic (X, q, d)-atoms supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$ such that

c3s2e5 (3.2.14)
$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \text{ in } \mathcal{S}'(\mathbb{R}^n)$$

and

c3s2e6 (3.2.15)
$$\|f\|_{H_X^A(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{1/\theta_0} \right\|_X .$$

First, this thesis try to find the desired function F. Note that a function $g \in L^1(\mathbb{R}^n)$ implies that $\widehat{g}$ is well defined in $\mathbb{R}^n$ (see, for instance, [31, (1.11)]), so does $\widehat{a_i}$ for any $i \in \mathbb{N}$. Moreover, from Lemmas 3.2.6 and 3.2.8 and from (3.2.15), it follows that, for any $x \in \mathbb{R}^n$,

c3s2e8 (3.2.16)
$$\sum_{i\in\mathbb{N}} |\lambda_i||\widehat{a_i}(x)| \lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \max\left\{ [\rho_*(x)]^{\frac{1}{q_0}-1}, [\rho_*(x)]^{\frac{1}{\theta_0}-1} \right\}$$

$$\lesssim \|f\|_{H_X^A(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1}\,,\,[\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\} < \infty.$$

Therefore, the function

c3s2e9 (3.2.17)
$$F(\cdot) := \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(\cdot)$$

is well defined pointwisely on $\mathbb{R}^n$ and, for any $x \in \mathbb{R}^n$,

$$|F(x)| \lesssim \|f\|_{H_X^A(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1}\,,\,[\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\},$$

which completes the proof of (3.2.4).

Second, this thesis shows the continuity of F on $\mathbb{R}^n$. If this thesis can prove that F is continuous on any compact subset of $\mathbb{R}^n$, then the continuity of F on $\mathbb{R}^n$ is obvious. Let $E \subset \mathbb{R}^n$ be any given compact set. Then there exists a positive constant K, depending only on A and E, such that $\rho_*(x) \le K$ holds for any $x \in E$. By this and (3.2.16), this thesis concludes that, for any $x \in E$,

$$\sum_{i\in\mathbb{N}} |\lambda_i||\widehat{a_i}(x)| \lesssim \max\left\{K^{\frac{1}{q_0}-1}, K^{\frac{1}{\theta_0}-1}\right\} \|f\|_{H_X^A(\mathbb{R}^n)} < \infty.$$

Thus, the summation $\sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(\cdot)$ converges uniformly on E. This, together with the fact that $\widehat{a_i}$ is continuous for any $i \in \mathbb{N}$, further implies that F is also continuous on E and hence on $\mathbb{R}^n$.

Finally, this thesis shows (3.2.3). By (3.2.14) and the continuity of the Fourier transform in $\mathcal{S}'(\mathbb{R}^n)$ (see, for instance, [31, Theorem 1.17]), this thesis obtains $\widehat{f} = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}$ in $\mathcal{S}'(\mathbb{R}^n)$. Thus, to prove (3.2.3), this thesis only need to show that

c3s2e10 (3.2.18)
$$F = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i} \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Indeed, from Lemma 3.2.6 and the definition of Schwartz functions [see (1.2.7)], this thesis deduces that, for any $i \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
|\langle \widehat{a_i}, \varphi\rangle| &= \left|\int_{\mathbb{R}^n} \widehat{a_i}(x)\varphi(x)\,dx\right| \\
&\le \sum_{k=1}^{\infty} \int_{(A^*)^{k+1}B_0^*\setminus(A^*)^k B_0^*} \max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1}\,,\,[\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\} |\varphi(x)|\,dx \\
&\quad + \|\varphi\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \sum_{k=1}^{\infty} b^{k+1} b^{k(\frac{1}{\theta_0}-1)} b^{-k(\lfloor\frac{1}{\theta_0}\rfloor+2)} + \|\varphi\|_{L^1(\mathbb{R}^n)} \le \sum_{k=1}^{\infty} b^{-k} + \|\varphi\|_{L^1(\mathbb{R}^n)},
\end{aligned}$$

where B_0^* is the unit dilated ball with respect to A^*. This further implies that there exists a positive constant C such that $|\langle \widehat{a_i}, \varphi\rangle| \le C$ holds uniformly for any $i \in \mathbb{N}$. Combining this and (3.2.15), this thesis obtains

$$\lim_{I\to\infty} \sum_{i=I+1}^{\infty} |\lambda_i||\langle \widehat{a_i}, \varphi\rangle| \lesssim \lim_{I\to\infty} \sum_{i=I+1}^{\infty} |\lambda_i| = 0.$$

Therefore, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle F, \varphi\rangle = \lim_{I\to\infty} \left\langle \sum_{i=1}^{I} \lambda_i \widehat{a_i},\, \varphi \right\rangle.$$

This finishes the proof of (3.2.18) and hence Theorem 3.2.1. □

c3s3 3.3 Hardy–Littlewood Inequalities on $H_X^A(\mathbb{R}^n)$

In this section, as applications of Theorem 3.2.1, this thesis first prove that the function F given in Theorem 3.2.1 has a higher order convergence at the origin (see Theorem 3.3.1). Then this thesis extends the Hardy–Littlewood inequality to the setting of anisotropic Hardy spaces associated with ball quasi-Banach function spaces (see Theorem 3.3.2).

c3s3t1 **Theorem 3.3.1.** *Let A, X, q_0, and ρ_* be the same as in Theorem 3.2.1. Then, for any $f \in H_X^A(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that $\widehat{f} = F$ in $\mathcal{S}'(\mathbb{R}^n)$ and*

c3s3e1 (3.3.1)
$$\lim_{|x|\to 0^+} \frac{F(x)}{[\rho_*(x)]^{\frac{1}{\theta_0}-1}} = 0.$$

Proof. Let $f \in H_X^A(\mathbb{R}^n)$ and q and d be the same as in Definition 3.2.3. Then, by Lemma 3.2.4 and Definition 3.2.3(ii), this thesis finds that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of anisotropic (X, q, d)-atoms supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$ such that $f = \sum_{i\in\mathbb{N}} \lambda_i a_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and

6.11.x2 (3.3.2)
$$\|f\|_{H_X^A(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X.$$

Moreover, from the proof of Theorem 3.2.1, it follows that, for any $x \in \mathbb{R}^n$,

c3s3e3 (3.3.3)
$$F(x) = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(x)$$

is continuous and satisfies that $\widehat{f} = F$ in $\mathcal{S}'(\mathbb{R}^n)$. Thus, to show the present theorem, this thesis only need to prove that (3.3.1) holds for F in (3.3.3). On the one hand, by an argument similar to that used in the proof of (3.2.13), this thesis concludes that, for any anisotropic (X, q, d)-atom a supported in $x_0 + B_{k_0}$ with some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$ and for any $x \in \mathbb{R}^n$ satisfying $\rho_*(x) \le b^{-k_0}$,

$$|\widehat{a}(x)| \lesssim \max\left\{b^{k_0[1-\frac{1}{q_0}+(d+1)\frac{\ln(\lambda_-)}{\ln b}]},\ b^{k_0[1-\frac{1}{\theta_0}+(d+1)\frac{\ln(\lambda_-)}{\ln b}]}\right\}[\rho_*(x)]^{(d+1)\frac{\ln(\lambda_-)}{\ln b}},$$

which, together with (2.2.1) and Lemma 3.2.7, further implies that $(d+1)\frac{\ln(\lambda_-)}{\ln b} > \frac{1}{\theta_0} - 1$ and

c3s3e4 (3.3.4)
$$\lim_{|x|\to 0^+} \frac{|\widehat{a}(x)|}{[\rho_*(x)]^{\frac{1}{\theta_0}-1}} = 0.$$

On the other hand, from (3.3.3), Lemmas 3.2.6, 3.2.7, and 3.2.8, and (3.3.2), this thesis deduces that, for any $x \in \mathbb{R}^n$ satisfying $|x| < 1$,

c3s3e5 (3.3.5)
$$\frac{|F(x)|}{[\rho_*(x)]^{\frac{1}{\theta_0}-1}} \le \sum_{i\in\mathbb{N}} |\lambda_i| \frac{|\widehat{a_i}(x)|}{[\rho_*(x)]^{\frac{1}{\theta_0}-1}} \lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \lesssim \|f\|_{H_X^A(\mathbb{R}^n)} < \infty.$$

Using this, the dominated convergence theorem, and (3.3.4), this thesis finds that

$$\lim_{|x|\to 0^+} \frac{F(x)}{[\rho_*(x)]^{\frac{1}{q_0}-1}} = 0,$$

which completes the proof of Theorem 3.3.1. □

As the other application of Theorem 3.2.1, this thesis extends the Hardy–Littlewood inequality to the setting of anisotropic Hardy spaces associated with ball quasi-Banach function spaces as follows.

c3s3t2 **Theorem 3.3.2.** *Let A, X, θ_0, and q_0 be the same as in Theorem 3.2.1. Then, for any $f \in H_X^A(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that $\widehat{f} = F$ in $\mathcal{S}'(\mathbb{R}^n)$ and*

c3s3e6 (3.3.6)
$$\left[\int_{\mathbb{R}^n} |F(x)|^{q_0} \min\left\{[\rho_*(x)]^{q_0-\frac{q_0}{\theta_0}-1},\ [\rho_*(x)]^{q_0-2}\right\} dx\right]^{\frac{1}{q_0}} \le C\|f\|_{H_X^A(\mathbb{R}^n)},$$

where C is a positive constant depending only on A and X.

Proof. Let p_0 and d be the same as in Definition 3.2.3, $q \in (\max\{p_0, 2\}, \infty]$, and $f \in H_X^A(\mathbb{R}^n)$. Then, by Lemma 3.2.4 and Definition 3.2.3, this thesis finds that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of (X, q, d)-atoms supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$ such that $f = \sum_{i\in\mathbb{N}} \lambda_i a_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and

c3s3e7 (3.3.7)
$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{1/\theta_0}\right\|_X \sim \|f\|_{H_X^A(\mathbb{R}^n)} < \infty.$$

By Theorem 3.2.1, this thesis finds that, to prove the present theorem, it suffices to show that (3.3.6) holds for F in (3.2.17). For this purpose, this thesis first prove that there exists a positive constant M such that, for any (X, q, d)-atom a, it holds that

c3s3e10
$$\left(\int_{\mathbb{R}^n}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{\theta_0}-\frac{1}{q_0}},\ [\rho_*(x)]^{1-\frac{2}{q_0}}\right\}\right]^{q_0}\,dx\right)^{\frac{1}{q_0}}\le M. \tag{3.3.8}$$

Without loss of generality, this thesis may assume that a is supported in $x_0+B_{i_0}$ with some $x_0\in\mathbb{R}^n$ and $i_0\in\mathbb{Z}$. Then it is easy to conclude that

6.11.z1
$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{\theta_0}-\frac{1}{q_0}},\ [\rho_*(x)]^{1-\frac{2}{q_0}}\right\}\right]^{q_0}\,dx\right\}^{\frac{1}{q_0}}\\
&\quad\lesssim\left\{\int_{(A^*)^{-i_0+1}B_0^*}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{\theta_0}-\frac{1}{q_0}},\ [\rho_*(x)]^{1-\frac{2}{q_0}}\right\}\right]^{q_0}\,dx\right\}^{\frac{1}{q_0}}\\
&\qquad+\left\{\int_{((A^*)^{-i_0+1}B_0^*)^\complement}\cdots\,dx\right\}^{\frac{1}{q_0}}\\
&\quad=:\mathrm{I}_1+\mathrm{I}_2,
\end{aligned}\tag{3.3.9}$$

where B_0^* is the unit dilated ball with respect to A^*. Let θ be a fixed positive constant such that

$$1-\frac{1}{q_0}+(d+1)\frac{\ln(\lambda_-)}{\ln b}-\theta\ge 1-\frac{1}{\theta_0}+(d+1)\frac{\ln(\lambda_-)}{\ln b}-\theta>0.$$

Using this and (3.2.13), this thesis finds that

6.11.z2
$$\begin{aligned}
\mathrm{I}_1&\lesssim b^{i_0[1+(d+1)\frac{\ln\lambda_-}{\ln b}]}\max\left\{b^{-\frac{i_0}{\theta_0}},\,b^{-\frac{i_0}{q_0}}\right\}\\
&\quad\times\left\{\int_{(A^*)^{-i_0+1}B_0^*}\left[\min\left\{[\rho_*(x)]^{1-\frac{1}{\theta_0}-\frac{1}{q_0}+(d+1)\frac{\ln\lambda_-}{\ln b}},\right.\right.\right.\\
&\quad\left.\left.\left.[\rho_*(x)]^{1-\frac{2}{q_0}+(d+1)\frac{\ln\lambda_-}{\ln b}}\right\}\right]^{q_0}\,dx\right\}^{\frac{1}{q_0}}\\
&\le b^{i_0[1+(d+1)\frac{\ln\lambda_-}{\ln b}]}\max\left\{b^{-\frac{i_0}{\theta_0}},\,b^{-\frac{i_0}{q_0}}\right\}\\
&\quad\times\min\left\{b^{-i_0[1-\frac{1}{\theta_0}+(d+1)\frac{\ln\lambda_-}{\ln b}-\theta]},\,b^{-i_0[1-\frac{1}{q_0}+(d+1)\frac{\ln\lambda_-}{\ln b}-\theta]}\right\}\\
&\quad\times\left\{\int_{(A^*)^{-i_0+1}B_0^*}[\rho_*(x)]^{\theta q_0-1}\,dx\right\}^{\frac{1}{q_0}}\\
&=b^{i_0\theta}\left[\sum_{k\in\mathbb{Z}\setminus\mathbb{N}}b^{-i_0+k}(b-1)b^{(-i_0+k)(\theta q_0-1)}\right]^{\frac{1}{q_0}}=\left(\frac{b-1}{1-b^{-\theta q_0}}\right)^{\frac{1}{q_0}}.
\end{aligned}\tag{3.3.10}$$

For I_2, by the Hölder inequality, the Plancherel theorem (see [31, Theorem 1.18]), $q_0 \in [\theta_0, 1]$, Definition 3.2.3(i)$_2$, and (3.2.2), this thesis obtains

$$\begin{aligned}
\mathrm{I}_2 &\le \left\{\int_{((A^*)^{-i_0+1}B_0^*)^\complement} \left|\widehat{a}(x)\right|^2\,dx\right\}^{\frac{1}{2}}\\
&\quad\times\left\{\int_{((A^*)^{-i_0+1}B_0^*)^\complement}\left[\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{q_0}},\,[\rho_*(x)]^{1-\frac{2}{q_0}}\right\}\right]^{\frac{2q_0}{2-q_0}}\,dx\right\}^{\frac{2-q_0}{2q_0}}\\
&\le \|a\|_{L^2(\mathbb{R}^n)}\left\{\sum_{k\in\mathbb{N}} b^{-i_0+k}(b-1)\left[\min\left\{b^{(-i_0+k)(1-\frac{1}{p_-}-\frac{1}{q_0})},\,b^{(-i_0+k)(1-\frac{2}{q_0})}\right\}\right]^{\frac{2q_0}{2-q_0}}\right\}^{\frac{2-q_0}{2q_0}}\\
&\le \|a\|_{L^2(\mathbb{R}^n)}\left\{b^{-i_0}\left[\min\left\{b^{-i_0(1-\frac{1}{p_-}-\frac{1}{q_0})},\,b^{-i_0(1-\frac{2}{q_0})}\right\}\right]^{\frac{2q_0}{2-q_0}}\right\}^{\frac{2-q_0}{2q_0}}\\
&\lesssim \max\left\{b^{i_0(\frac{1}{2}-\frac{1}{p_-})},\,b^{i_0(\frac{1}{2}-\frac{1}{q_0})}\right\}\min\left\{b^{-i_0(\frac{1}{2}-\frac{1}{p_-})},\,b^{-i_0(\frac{1}{2}-\frac{1}{q_0})}\right\}=1,
\end{aligned}$$

which, together with (3.3.9) and (3.3.10), further implies (3.3.8).

Next, this thesis proves (3.3.6). From Lemma 2.1.8, Definition 1.2.6, (3.2.1), Definition 1.2.4(ii), an argument similar to that used in the proof of Lemma 3.2.8, and (3.3.7), this thesis deduces that

$$\left(\sum_{i=1}^{\infty}|\lambda_i|^{q_0}\right)^{\frac{1}{q_0}} \lesssim \|f\|_{H_X^A(\mathbb{R}^n)}.$$

By this, (3.2.17), $q_0 \in [\theta_0, 1]$, Lemma 2.1.8, the Fatou lemma, and (3.3.8), this thesis concludes that

$$\begin{aligned}
&\left[\int_{\mathbb{R}^n}|F(x)|^{q_0}\min\left\{[\rho_*(x)]^{q_0-\frac{q_0}{\theta_0}-1},\,[\rho_*(x)]^{q_0-2}\right\}\,dx\right]^{\frac{1}{q_0}}\\
&\quad\le\left\{\sum_{i\in\mathbb{N}}|\lambda_i|^{q_0}\int_{\mathbb{R}^n}\left[|\widehat{a_i}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{\theta_0}-\frac{1}{q_0}},\,[\rho_*(x)]^{1-\frac{2}{q_0}}\right\}\right]^{q_0}\,dx\right\}^{\frac{1}{q_0}}\\
&\quad\lesssim M\left(\sum_{i\in\mathbb{N}}|\lambda_i|^{q_0}\right)^{\frac{1}{q_0}}\lesssim\|f\|_{H_X^A(\mathbb{R}^n)}.
\end{aligned}$$

This finishes the proof of Theorem 3.3.2. □

6.11rem **Remark 3.3.3.** (i) If $A := 2\,I_{n\times n}$, then Theorems 3.3.1 and 3.3.2 were obtained, respectively, in [46, Theorems 2.2 and 2.3].

(ii) Let A be a dilation and $p \in (0,1)$. Then, by Remark 3.2.2(ii), this thesis finds that $L^p(\mathbb{R}^n)$ satisfies all assumptions of Theorems 3.3.1 and 3.3.2 with $X := L^p(\mathbb{R}^n)$. In this case, Theorems 3.3.1 and 3.3.2 were obtained, respectively, in [9, Corollaries 6 and 8].

Chapter 4

Real-Variable Characterizations of $\mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)$

c4

4.1 Equivalent Characterizations of $\mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)$

c4s1

In this section, applying the dual theorem obtained in Section 2, this thesis establishes several equivalent characterizations for the anisotropic ball Campanato function space $\mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)$. This plays an important role in establishing the Carleson measure characterization of $\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)$ in Section 4.2 below.

s3t1 **Theorem 4.1.1.** *Let A, X, q, d, and θ_0 be the same as in Corollary 2.2.7 and*

2.19.y2

$$\varepsilon \in \left(\frac{\ln b}{\ln(\lambda_-)}\left[\frac{2}{s}+d\frac{\ln(\lambda_+)}{\ln b}\right],\infty\right) \tag{4.1.1}$$

for some $s \in (0,\theta_0)$. Then the following statements are mutually equivalent:

(i) $f \in \mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)$;

(ii) $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ *and*

2.19.y1

$$\begin{aligned}\|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)} := \sup & \left\|\left\{\sum_{i=1}^{m}\left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \\ & \times\sum_{j=1}^{m}\frac{\lambda_j|x_j+B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\ & \times\int_{\mathbb{R}^n}\frac{b^{\varepsilon l_j\frac{\ln(\lambda_-)}{\ln b}}|f(x)-P^d_{x_j+B_{l_j}}f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]}+[\rho(x-x_j)]^{1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}}}\,dx\end{aligned} \tag{4.1.2}$$

$$< \infty,$$

where the supremum is taken over all $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$, *with both* $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ *and* $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, *and* $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$.

Moreover, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}$$

with the positive equivalence constants independent of f.

To show Theorem 4.1.1, this thesis needs the following technical lemma, which is a direct inference of the pointwise estimate $\mathbf{1}_{x_j+B_{k_j+\ell}} \le b^\ell \mathcal{M}(\mathbf{1}_{x_j+B_{k_j}})$, for any $\ell \in \mathbb{Z}_+$, sequence $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$, and sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$; this thesis omits the details here.

s312 **Lemma 4.1.2.** *Let* X *be a ball quasi-Banach function space satisfying Assumption 1.2.10 with* $p_- \in (0,\infty)$, $\ell \in \mathbb{Z}_+$, *and* $s \in (0, \min\{p_-, 1\})$. *Then there exists a positive constant* C, *independent of both* ℓ *and* s, *such that, for any sequence* $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ *and any sequence* $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$,

$$\left\|\sum_{j\in\mathbb{N}} \mathbf{1}_{x_j+B_{k_j+\ell}}\right\|_X \le C b^{\frac{\ell}{s}} \left\|\sum_{j\in\mathbb{N}} \mathbf{1}_{x_j+B_{k_j}}\right\|_X,$$

where, for any $j \in \mathbb{N}$, B_{k_j} *is the same as in* (1.2.2).

Now, this thesis shows Theorem 4.1.1.

Proof of Theorem 4.1.1. According to Corollary 2.2.7, to prove the present theorem, this thesis only need to show that, for any $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

2.18.x1
$$\|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}. \tag{4.1.3}$$

This thesis first prove

2.17.x1
$$\|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}. \tag{4.1.4}$$

Indeed, by Definition 1.2.3, this thesis finds that, for any $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$, $\varepsilon \in (0,\infty)$, and $j \in \{1,2,\ldots,m\}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x-x_j)]^{1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}}}\,dx \\
&\quad \ge \int_{x_j+B_{l_j}} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x-x_j)]^{1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}}}\,dx
\end{aligned}$$

$$\sim \int_{x_j+B_{l_j}} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]}}\, dx$$
$$= \fint_{x_j+B_{l_j}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx,$$

which, together with Definition 2.1.3 and (4.1.2), further implies (4.1.4).

Conversely, Define

2.18.x2 (4.1.5)
$$\mathrm{I} := \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k=0}^{\infty} b^{-k[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} \int_{x_j+B_{l_j+k+1}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx. \tag{4.1.5}$$

This thesis has

$$\begin{aligned}\mathrm{I} &\lesssim \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k\in\mathbb{N}} b^{-k[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} \int_{x_j+B_{l_j+k}} \left| f(x) - P^d_{x_j+B_{l_j+k}} f(x) \right| dx \\ &\quad + \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k\in\mathbb{N}} b^{-k[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} \\ &\quad \times \int_{x_j+B_{l_j+k}} \left| P^d_{x_j+B_{l_j+k}} f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx.\end{aligned}$$

From Definitions 1.2.3 and 2.1.3, this thesis deduces that, for any $m \in \mathbb{N}$, $\{x_j+B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$,

s3e1 (4.1.6)
$$\begin{aligned}&\sum_{j=1}^{m} \frac{\lambda_j |x_j+B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \int_{\mathbb{R}^n} \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x-x_j)]^{1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}}}\, dx \\ &\quad = \sum_{j=1}^{m} \frac{\lambda_j |x_j+B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \left[\int_{x_j+B_{l_j}} + \sum_{k=0}^{\infty} \int_{x_j+B_{l_j+k+1}\setminus x_j+B_{l_j+k}} \right] \\ &\quad \times \frac{b^{\varepsilon l_j \frac{\ln(\lambda_-)}{\ln b}} |f(x) - P^d_{x_j+B_{l_j}} f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + [\rho(x-x_j)]^{1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}}}\, dx \\ &\quad \le \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \int_{x_j+B_{l_j}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx \\ &\quad + \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \sum_{k=0}^{\infty} b^{-k[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} \\ &\quad \times \int_{x_j+B_{l_j+k+1}} \left| f(x) - P^d_{x_j+B_{l_j}} f(x) \right| dx \\ &\quad \le \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}} \right\}^{\frac{1}{\theta_0}} \right\|_X \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} + \mathrm{I},\end{aligned} \tag{4.1.6}$$

Note that, on the one hand, by the definition of minimizing polynomials, (1.2.5), [53, Lemma 2.19], and Lemma 2.2.5, this thesis finds that, for any $k \in \mathbb{N}$ and $x \in x_j + B_{l_j+k+1}$,

2.19.x1 (4.1.7)
$$\begin{aligned}
&\left|P^d_{x_j+B_{l_j+k}} f(x) - P^d_{x_j+B_{l_j}} f(x)\right| \\
&\quad \le \sum_{\nu=1}^{k} \left|P^d_{x_j+B_{l_j+\nu}} f(x) - P^d_{x_j+B_{l_j+\nu-1}} f(x)\right| \\
&\quad = \sum_{\nu=1}^{k} \left|P^d_{x_j+B_{l_j+\nu-1}} \left(f - P^d_{x_j+B_{l_j+\nu}} f\right)(x)\right| \\
&\quad \le \sum_{\nu=1}^{k} \left\|P^d_{x_j+B_{l_j+\nu-1}} \left(f - P^d_{x_j+B_{l_j+\nu}} f\right)\right\|_{L^\infty(B(x_j,\lambda_+^{l_j+k}))} \\
&\quad \lesssim \sum_{\nu=1}^{k} \left(\frac{\lambda_+^{l_j+k}}{\lambda_-^{l_j+v-1}}\right)^d \left\|P^d_{x_j+B_{l_j+\nu-1}} \left(f - P^d_{x_j+B_{l_j+\nu}} f\right)\right\|_{L^\infty(B(x_j,\lambda_-^{l_j+\nu-1}))} \\
&\quad \lesssim \lambda_+^{kd} \sum_{\nu=1}^{k} \frac{1}{|x_j + B_{l_j+\nu-1}|} \int_{x_j+B_{l_j+\nu}} \left|f(y) - P^d_{x_j+B_{l_j+\nu}} f(y)\right| \, dy;
\end{aligned}$$

on the other hand, from Definition 1.2.4(ii), the fact that $s \in (0, \theta_0)$, and Lemma 4.1.2, this thesis infers that, for any $j \in \{1, 2, \dots, m\}$,

2.19.x3 (4.1.8)
$$\frac{1}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \lesssim b^{\frac{k}{s}} \frac{1}{\|\mathbf{1}_{x_j+B_{l_j+k}}\|_X}$$

and, for any $k \in \mathbb{N}$,

2.19.x2 (4.1.9)
$$\begin{aligned}
&\left\|\left\{\sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+k}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+k}}\right\}^{\frac{1}{\theta_0}}\right\|_X \\
&\quad \le \left\|\left\{\sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+k}}\right\}^{\frac{1}{\theta_0}}\right\|_X \\
&\quad \lesssim b^{\frac{k}{s}} \left\|\left\{\sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X .
\end{aligned}$$

Combining (4.1.5), (4.1.7), (4.1.8), (4.1.9), and Lemma 4.1.2, this thesis concludes that

$$\left\|\left\{\sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \times \mathrm{I}$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+k}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+k}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \\
&\quad \times \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j+k}}\|_X} \int_{x_j+B_{l_j+k}} \left| f(x) - P^d_{x_j+B_{l_j+k}} f(x) \right| dx \\
&\quad + \sum_{k\in\mathbb{N}} \left(\frac{\lambda_+^d}{\lambda_-^\varepsilon} b \right)^k \sum_{\nu=1}^{k} b^{\nu(\frac{2}{s}-1)} \left\| \left\{ \sum_{i=1}^{m} \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+\nu}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+\nu}} \right\}^{\frac{1}{\theta_0}} \right\|_X^{-1} \\
&\quad \times \sum_{j=1}^{m} \frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j+\nu}}\|_X} \int_{x_j+B_{l_j+\nu}} \left| f(y) - P^d_{x_j+B_{l_j+\nu}} f(y) \right| dy \\
&\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + \sum_{k\in\mathbb{N}} \left(\frac{\lambda_+^d}{\lambda_-^\varepsilon} b \right)^k \sum_{\nu=1}^{k} b^{\nu(\frac{2}{s}-1)} \right\} \\
&\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + \sum_{k\in\mathbb{N}} \left(\frac{\lambda_+^d}{\lambda_-^\varepsilon} b \right)^k b^{(\frac{2}{s}-1)k} \right\} \\
&\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \left\{ \sum_{k\in\mathbb{N}} b^{-k[1-\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]} + \sum_{k\in\mathbb{N}} b^{-k[\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}-d\frac{\ln(\lambda_+)}{\ln b}]} \right\} \\
&\sim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \sum_{k\in\mathbb{N}} b^{-k[\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}-d\frac{\ln(\lambda_+)}{\ln b}]},
\end{aligned}
$$

which, together with (4.1.6), (4.1.2), (4.1.1), and the arbitrariness of $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0,\infty)$, further implies that

$$
\begin{aligned}
\|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)} &\lesssim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \sum_{k\in\mathbb{N}} b^{-k[\frac{2}{s}+\varepsilon\frac{\ln(\lambda_-)}{\ln b}-d\frac{\ln(\lambda_+)}{\ln b}]} \\
&\sim \|f\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.
\end{aligned}
$$

This, combined with (4.1.4), proves (4.1.3) and hence finishes the proof of Theorem 4.1.1. □

This thesis can obtain one more equivalent characterization of $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$ as follows, whose proof is a slight modification of Theorem 4.1.1; this thesis omits the details.

s3t2 **Theorem 4.1.3.** *If A, X, q, d, θ_0, and ε are the same as in Theorem 4.1.1, then the conclusion of Theorem 4.1.1 with m replaced by ∞ still holds true, where the supremum therein is taken over all $\{x_j + B_{l_j}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ with both $\{x_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{l_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ and*

$\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ *satisfying*

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_j+B_{l_j}}\right\}^{\frac{1}{\theta_0}}\right\|_X \in (0,\infty).$$

3.24.x2 **Remark 4.1.4.** If $A := 2\,I_{n\times n}$, then Theorems 4.1.1 and 4.1.3 were obtained in [103, Theorems 4.1 and 4.4], respectively.

c4s2 4.2 Carleson Measure Characterization of $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$

In this section, applying the results obtained in previous sections, this thesis establishes the Carleson measure characterization of the anisotropic ball Campanato function space $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$. To this end, this thesis first introduce the following *anisotropic X–Carleson measure*.

Definition 4.2.1. Let A be a dilation, X a ball quasi-Banach function space and $s\in(0,\infty)$. A Borel measure $d\mu$ on $\mathbb{R}^n\times\mathbb{Z}$ is called an *anisotropic X–Carleson measure* if

$$\begin{aligned}\|d\mu\|_X^{A,s} := \sup &\left\|\left\{\sum_{i=1}^m\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\\ &\times\sum_{j=1}^m\left\{\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\right\}\\ &<\infty,\end{aligned}$$

where the supremum is taken over all $m\in\mathbb{N}$, $\{B^{(j)}\}_{j=1}^m\subset\mathcal{B}$, and $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$, and, for any $j\in\{1,\dots,m\}$, $\widehat{B^{(j)}}$ denotes the *tent* over $B^{(j)}$, that is,

hatB (4.2.1)
$$\widehat{B^{(j)}} := \left\{(y,k)\in\mathbb{R}^n\times\mathbb{Z}:\ y+B_k\subset B^{(j)}\right\}.$$

For the anisotropic X–Carleson measure, this thesis has the following equivalent characterization.

s5p1 **Proposition 4.2.2.** *Let A be a dilation, X a ball quasi-Banach function space, $d\mu$ a Borel measure on $\mathbb{R}^n\times\mathbb{Z}$, $s\in(0,\infty)$, and*

$$\begin{aligned}\widetilde{\|d\mu\|}_X^{A,s} := \sup &\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\\ &\times\sum_{j\in\mathbb{N}}\left\{\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\right\},\end{aligned}$$

where the supremum is taken over all $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ *and* $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ *satisfying*

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X\in(0,\infty).$$

Then $\widetilde{\|d\mu\|}_X^{A,s}=\|d\mu\|_X^{A,s}$.

Proof. Let $d\mu$ be a Borel measure on $\mathbb{R}^n\times\mathbb{Z}$. Obviously, $\|d\mu\|_X^{A,s}\le\widetilde{\|d\mu\|}_X^{A,s}$. This thesis next shows

s5e1 (4.2.2) $$\widetilde{\|d\mu\|}_X^{A,s}\le\|d\mu\|_X^{A,s}.$$

Indeed, for any $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset[0,\infty)$ as in the present proposition, by Definition 1.2.4(iii), this thesis finds that

$$\begin{aligned}&\lim_{m\to\infty}\left\|\left\{\sum_{i=1}^{m}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j=1}^{m}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\\&=\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j\in\mathbb{N}}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}.\end{aligned}$$

Therefore, for any given $\varepsilon\in(0,\infty)$, there exists an $m_0\in\mathbb{N}$ such that $\sum_{j=1}^{m_0}\lambda_j\neq0$ and

$$\begin{aligned}&\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j\in\mathbb{N}}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\\&<\left\|\left\{\sum_{i=1}^{m_0}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\sum_{j=1}^{m_0}\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}+\varepsilon\\&\le\|d\mu\|_X^{A,s}+\varepsilon.\end{aligned}$$

Combining this, the arbitrariness of both $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ and $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ as in the present proposition, and $\varepsilon\in(0,\infty)$, this thesis further obtain (4.2.2) and hence complete the proof of Proposition 4.2.2. □

In what follows, for any given $k\in\mathbb{Z}$, define

$$\delta_k(j):=\begin{cases}1 & \text{when } j=k,\\ 0 & \text{when } j\neq k.\end{cases}$$

Next, this thesis state the main theorem of this section as follows.

s5t1 **Theorem 4.2.3.** *Let A, X, d, and θ_0 be the same as in Definition 2.2.1, $p_0 \in (\theta_0, 2)$, and $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial real-valued function satisfying* (3.1.1) *and* (3.1.2).

(i) *If $h \in \mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$, then, for any $(x,k) \in \mathbb{R}^n \times \mathbb{Z}$, $d\mu(x,k) := \sum_{\ell\in\mathbb{Z}} |\phi_\ell * h(x)|^2 dx\, \delta_\ell(k)$ is an X–Carleson measure on $\mathbb{R}^n \times \mathbb{Z}$; moreover, there exists a positive constant C, independent of h, such that*

$$\|d\mu\|_X^{A,\theta_0} \le C\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.$$

(ii) *If $h \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and, for any $(x,k) \in \mathbb{R}^n \times \mathbb{Z}$, $d\mu(x,k) := \sum_{\ell\in\mathbb{Z}} |\phi_\ell * h(x)|^2 dx\, \delta_\ell(k)$ is an X–Carleson measure on $\mathbb{R}^n \times \mathbb{Z}$, then $h \in \mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ and, moreover, there exists a positive constant C, independent of h, such that*

$$\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \le C \, \|d\mu\|_X^{A,\theta_0}.$$

Remark 4.2.4. (i) Note that, if X is a concave ball quasi-Banach function space, then, by Proposition 2.1.9, Theorem 4.2.3 gives the Carleson measure characterization of $\mathcal{L}^A_{X,1,d}(\mathbb{R}^n)$.

(ii) If $A := 2\, I_{n\times n}$, then Theorem 4.2.3 was obtained in [103, Theorem 5.3].

To prove Theorem 4.2.3, this thesis needs the anisotropic tent space associated with ball quasi-Banach function space and its atomic decomposition. This thesis first recall the following concept.

defcone **Definition 4.2.5.** Let A be a dilation and, for any $x \in \mathbb{R}^n$, let

$$\Gamma(x) := \{(y,k) \in \mathbb{R}^n \times \mathbb{Z} : \ y \in x + B_k\},$$

which is called the *cone* of aperture 1 with vertex $x \in \mathbb{R}^n$.

Let $\alpha \in (0,\infty)$. For any measurable function $F : \ \mathbb{R}^n \times \mathbb{Z} \to \mathbb{C}$ and $x \in \mathbb{R}^n$, define

$$\mathscr{A}(F)(x) := \left[\sum_{\ell\in\mathbb{Z}} b^{-\ell} \int_{\{y\in\mathbb{R}^n: \ (y,\ell)\in\Gamma(x)\}} |F(y,\ell)|^2 \, dy\right]^{\frac{1}{2}},$$

where $\Gamma(x)$ is the same as in Definition 4.2.5. A measurable function F on $\mathbb{R}^n \times \mathbb{Z}$ is said to belong to the *anisotropic tent space* $T_2^{A,p}(\mathbb{R}^n \times \mathbb{Z})$, with $p \in (0,\infty)$, if

$$\|F\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})} := \|\mathscr{A}(F)\|_{L^p(\mathbb{R}^n)} < \infty.$$

For any given ball quasi-Banach function space X, the *anisotropic X-tent space* $T_X^A(\mathbb{R}^n\times\mathbb{Z})$ is defined to be the set of all the measurable functions F on $\mathbb{R}^n \times \mathbb{Z}$ such that $\mathscr{A}(F) \in X$ and naturally equipped with the quasi-norm $\|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})} := \|\mathscr{A}(F)\|_X$.

This thesis next give the definition of anisotropic (T_X, p)-atoms.

s5d1 **Definition 4.2.6.** Let $p \in (1,\infty)$, A be a dilation, and X a ball quasi-Banach function space. A measurable function $a:\ \mathbb{R}^n \times \mathbb{Z} \to \mathbb{C}$ is called an *anisotropic* (T_X, p)*-atom* if there exists a ball $B \subset \mathcal{B}$ such that

(i) $\operatorname{supp} a := \{(x,k) \in \mathbb{R}^n \times \mathbb{Z}:\ a(x,k) \neq 0\} \subset \widehat{B}$, where $\widehat{B}$ is the same as in (4.2.1) with $B^{(j)}$ replaced by B.

(ii) $\|a\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})} \le |B|^{1/p}/\|\mathbf{1}_B\|_X$.

Moreover, if a is an anisotropic (T_X, p)-atom for any $p \in (1,\infty)$, then a is called an *anisotropic* (T_X,∞)*-atom.*

This thesis has the following atomic decomposition on the anisotropic X-tent space $T_X^A(\mathbb{R}^n \times \mathbb{Z})$.

s5l1 **Lemma 4.2.7.** *Let A, X, and θ_0 be the same as in Definition 2.2.1 and $F:\ \mathbb{R}^n\times\mathbb{Z} \to \mathbb{C}$ a measurable function. If $F \in T_X^A(\mathbb{R}^n \times \mathbb{Z})$, then there exists a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset [0,\infty)$, a sequence $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$, and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of anisotropic (T_X,∞)-atoms supported, respectively, in $\{\widehat{B^{(j)}}\}_{j\in\mathbb{N}}$ such that, for almost every $(x,k) \in \mathbb{R}^n \times \mathbb{Z}$,*

$$F(x,k) = \sum_{j\in\mathbb{N}} \lambda_j A_j(x,k),\ |F(x,k)| = \sum_{j\in\mathbb{N}} \lambda_j |A_j(x,k)|$$

pointwisely, and

s5e2 (4.2.3)
$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left(\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right)^{\theta_0} \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{\theta_0}} \right\|_X \lesssim \|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})},$$

where the implicit positive constant is independent of F.

Proof. For any $j \in \mathbb{Z}$, let

$$O_j := \left\{ x \in \mathbb{R}^n :\ \mathscr{A}(F)(x) > 2^j \right\},$$

$F_j := (O_j)^{\complement}$, and, for any given $\gamma \in (0,1)$,

$$(O_j)^*_\gamma := \left\{ x \in \mathbb{R}^n : \mathcal{M}(\mathbf{1}_{O_j})(x) > 1-\gamma \right\}.$$

Then, by an argument similar to that used in the proof of [33, (1.14)], this thesis finds that

$$\operatorname{supp} F \subset \left[\bigcup_{j\in\mathbb{Z}} \widehat{(O_j)^*_\gamma} \cup E \right],$$

where $E \subset \mathbb{R}^n \times \mathbb{Z}$ satisfies that

$$\sum_{\ell \in \mathbb{Z}} \int_{\{y \in \mathbb{R}^n:\ (y,\ell) \in E\}} dy = 0.$$

Moreover, applying [33, (1.15)], this thesis concludes that, for any $j \in \mathbb{Z}$, there exists an integer $N_j \in \mathbb{N} \cup \{\infty\}$, $\{x_k^{(j)}\}_{k=1}^{N_j} \subset (O_j)_\gamma^*$, and $\{l_k\}_{k=1}^{N_j} \subset \mathbb{Z}$ such that $\{x_k^{(j)} + B_{l_k}^{(j)}\}_{k=1}^{N_j}$ has the finite intersection property and

s5l1e2 (4.2.4)

$$\begin{aligned}
(O_j)_\gamma^* &= \bigcup_{k=1}^{N_j} \left[x_k^{(j)} + B_{l_k}^{(j)}\right] \\
&= \left[x_1^{(j)} + B_{l_1}^{(j)}\right] \cup \left\{\left[x_2^{(j)} + B_{l_2}^{(j)}\right] \setminus \left[x_1^{(j)} + B_{l_1}^{(j)}\right]\right\} \cup \cdots \\
&\quad \cup \left\{\left[x_{N_j}^{(j)} + B_{l_{N_j}}^{(j)}\right] \setminus \bigcup_{i=1}^{N_j-1} \left[x_i^{(j)} + B_{l_i}^{(j)}\right]\right\} \\
&=: \bigcup_{k=1}^{N_j} B_{j,k}.
\end{aligned} \tag{4.2.4}$$

Notice that, for any $j \in \mathbb{Z}$, $\{B_{j,k}\}_{k=1}^{N_j}$ are mutually disjoint. Thus, $\widehat{(O_j)_\gamma^*} = \bigcup_{k=1}^{N_j} \widehat{B_{j,k}}$. For any $j \in \mathbb{Z}$ and $k \in \{1, \ldots, N_j\}$, let

s5l1e4 (4.2.5)

$$C_{j,k} := \widehat{B_{j,k}} \cap \left[\widehat{(O_j)_\gamma^*} \setminus \widehat{(O_{j+1})_\gamma^*}\right],\ A_{j,k} := 2^{-j} \left\| \mathbf{1}_{x_k^{(j)} + B_{l_k}^{(j)}} \right\|_X^{-1} F \mathbf{1}_{C_{j,k}}, \tag{4.2.5}$$

and $\lambda_{j,k} := 2^j \|\mathbf{1}_{x_k^{(j)} + B_{l_k}^{(j)}}\|_X$. Therefore, from (4.2.4), it follows that

$$F = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{N_j} \lambda_{j,k} A_{j,k} \ \text{ and } \ |F| = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{N_j} \lambda_{j,k} |A_{j,k}|$$

almost everywhere on $\mathbb{R}^n \times \mathbb{Z}$. This thesis now show that, for any $j \in \mathbb{Z}$ and $k \in \{1, \ldots, N_j\}$, $A_{j,k}$ is an anisotropic (T_X^A, ∞)-atom supported in $\widehat{x_k^{(j)} + B_{l_k}^{(j)}}$ up to a harmless constant multiple. Obviously,

$$\operatorname{supp} A_{j,k} \subset C_{j,k} \subset \widehat{B_{j,k}} \subset \widehat{x_k^{(j)} + B_{l_k}^{(j)}}.$$

In addition, let $p \in (1, \infty)$ and $h \in T_2^{A,p'}(\mathbb{R}^n \times \mathbb{Z})$ satisfy $\|h\|_{T_2^{A,p'}(\mathbb{R}^n \times \mathbb{Z})} \leq 1$. Notice that

$$C_{j,k} \subset \widehat{(O_{j+1})_\gamma^*}^{\complement} = \bigcup_{x \in (O_{j+1})_\gamma^{*\,\complement}} \Gamma(x).$$

Applying this, [33, Lemma 1.3], the Hölder inequality, and (4.2.5), this thesis finds that

$$\begin{aligned}
|\langle A_{j,k}, h\rangle| &= \left|\sum_{\ell\in\mathbb{Z}} \int_{\mathbb{R}^n} A_{j,k}(y,\ell)h(y,\ell)\mathbf{1}_{C_{j,k}}(y,\ell)\,dy\right| \\
&\le \sum_{\ell\in\mathbb{Z}} \int_{(y,\ell)\in\bigcup_{x\in(O_{j+1})^{*}_{\gamma}{}^{\complement}}\Gamma(x)} |A_{j,k}(y,\ell)h(y,\ell)|\,dy\,\delta_i(\ell) \\
&\lesssim \int_{(O_{j+1})^{\complement}} \left[\sum_{\ell\in\mathbb{Z}} \int_{\{y\in\mathbb{R}^n:\ (y,\ell)\in\Gamma(x)\}} b^{-\ell}\,|A_{j,k}(y,\ell)h(y,\ell)|\,dy\right] dx \\
&\le \int_{(O_{j+1})^{\complement}} \mathscr{A}\left(A_{j,k}\right)(x)\mathscr{A}(h)(x)\,dx \\
&\le \left\{\int_{(O_{j+1})^{\complement}} \left[\mathscr{A}\left(A_{j,k}\right)(x)\right]^p\,dx\right\}^{\frac{1}{p}} \left\{\int_{(O_{j+1})^{\complement}} [\mathscr{A}(h)(x)]^{p'}\,dx\right\}^{\frac{1}{p'}} \\
&\le 2^{-j}\left\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\right\|_X^{-1} \left\{\int_{(x_k^{(j)}+B_{l_k}^{(j)})\cap(O_{j+1})^{\complement}} [\mathscr{A}(F)(x)]^p\,dx\right\}^{\frac{1}{p}} \\
&\quad\times \|h\|_{T_2^{A,p'}(\mathbb{R}^n\times\mathbb{Z})} \\
&\lesssim \frac{|x_k^{(j)}+B_{l_k}^{(j)}|^{\frac{1}{p}}}{\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\|_X},
\end{aligned}$$

which, combined with $(T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z}))^* = T_2^{A,p'}(\mathbb{R}^n\times\mathbb{Z})$ (see [24, Theorem 2]), further implies that

$$\|A_{j,k}\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})} \lesssim \frac{|x_k^{(j)}+B_{l_k}^{(j)}|^{\frac{1}{p}}}{\|\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\|_X}.$$

Using this, this thesis finds that, for any $j\in\mathbb{Z}$ and $k\in\{1,\ldots,N_j\}$, $A_{j,k}$ is an anisotropic (T_X^A,p)-atom up to a harmless constant multiple for any $p\in(1,\infty)$. Thus, for any $j\in\mathbb{Z}$ and $k\in\{1,\ldots,N_j\}$, $A_{j,k}$ is an anisotropic (T_X^A,∞)-atom up to a harmless constant multiple.

This thesis next prove (4.2.3). To achieve this, from (4.2.4), the finite intersection property of $\{x_k^{(j)}+B_{l_k}^{(j)}\}_{k=1}^{N_j}$, the estimate that $\mathbf{1}_{(O_j)^*_\gamma} \lesssim [\mathcal{M}(\mathbf{1}_{O_j})]^{\frac{1}{\theta_0}}$, and Assumption 1.2.10, this thesis deduces that

$$\left\|\left\{\sum_{j\in\mathbb{Z}}\sum_{k=1}^{N_j}\left[\frac{\lambda_{j,k}}{||\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}||_X}\right]^{\theta_0}\mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X$$

$$
\begin{aligned}
&= \left\| \left\{ \sum_{j\in\mathbb{Z}} \sum_{k=1}^{N_j} \left[2^j \mathbf{1}_{x_k^{(j)}+B_{l_k}^{(j)}} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} \left[2^j \mathbf{1}_{(O_j)_\gamma^*} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X \lesssim \left\| \sum_{j\in\mathbb{Z}} \left\{ 2^j \left[\mathcal{M}\left(\mathbf{1}_{O_j}\right) \right]^{\frac{1}{\theta_0}} \right\}^{\theta_0} \right\|_{X^{\frac{1}{\theta_0}}}^{\frac{1}{\theta_0}} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} \left(2^j \mathbf{1}_{O_j} \right)^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X \sim \left\| \left\{ \sum_{j\in\mathbb{Z}} \left(2^j \mathbf{1}_{O_j\setminus O_{j+1}} \right)^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X \\
&\leq \left\| \mathscr{A}(F) \left[\sum_{j\in\mathbb{Z}} \mathbf{1}_{O_j\setminus O_{j+1}} \right]^{\frac{1}{\theta_0}} \right\|_X = \|\mathscr{A}(F)\|_X = \|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})}.
\end{aligned}
$$

This further implies that (4.2.3) holds true and hence finishes the proof of Lemma 4.2.7. □

This thesis now prove Theorem 4.2.3.

Proof of Theorem 4.2.3. This thesis first show (i). To this end, let $h \in \mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ and $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with $m \in \mathbb{N}$, $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$, and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$. Then this thesis easily find that, for any $j \in \{1, \ldots, m\}$,

s5e3 (4.2.6)
$$
\begin{aligned}
h &= P^d_{x_j+B_{l_j}} h + \left(h - P^d_{x_j+B_{l_j}} h \right) \mathbf{1}_{x_j+B_{l_j+\tau}} + \left(h - P^d_{x_j+B_{l_j}} h \right) \mathbf{1}_{(x_j+B_{l_j+\tau})^\complement} \\
&=: h_j^{(1)} + h_j^{(2)} + h_j^{(3)},
\end{aligned}
$$

where τ is the same as in (1.2.4). For $h_j^{(1)}$, by the fact that $\int_{\mathbb{R}^n} \phi(x) x^\alpha \, dx = 0$ for any $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq d$, this thesis concludes that, for any $k \in \mathbb{Z}$, $\phi_k * h_j^{(1)} \equiv 0$ and hence

s5e4 (4.2.7)
$$
\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left| \phi_k * h_j^{(1)}(x) \right|^2 dx = 0.
$$

For $h_j^{(2)}$, from the Tonelli theorem and the boundedness on $L^2(\mathbb{R}^n)$ of the anisotropic g-function

$$
g(h_j^{(2)}) := \left[\sum_{k\in\mathbb{Z}} \left| \phi_k * h_j^{(2)} \right|^2 \right]^{\frac{1}{2}}
$$

(see, for instance, [47, Theorem 6.3]), this thesis infers that

s5e5 (4.2.8)
$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left| \phi_k * h_j^{(2)}(x) \right|^2 dx \\
&\quad \leq \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left| \phi_k * h_j^{(2)}(x) \right|^2 dx \lesssim \left\| h_j^{(2)} \right\|^2_{L^2(\mathbb{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j}} h(x)\right|^2 \, dx \\
&\le \int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j+\tau}} h(x)\right|^2 \, dx \\
&\quad + \int_{x_j+B_{l_j+\tau}} \left|P^d_{x_j+B_{l_j+\tau}} h(x) - P^d_{x_j+B_{l_j}} h(x)\right|^2 \, dx.
\end{aligned}
$$

In addition, using Lemma 2.2.5, this thesis obtains, for any $x \in x_j + B_{l_j+\tau}$,

$$
\begin{aligned}
&\left|P^d_{x_j+B_{l_j+\tau}} h(x) - P^d_{x_j+B_{l_j}} h(x)\right| \\
&\quad = \left|P^d_{x_j+B_{l_j+\tau}} \left(h - P^d_{x_j+B_{l_j}} h\right)(x)\right| \\
&\quad \lesssim \frac{1}{|x_j + B_{l_j}|} \int_{x_j+B_{l_j+\tau}} \left|h(y) - P^d_{x_j+B_{l_j+\tau}} h(y)\right| \, dy.
\end{aligned}
$$

Thus, combining this with (4.2.8), Lemma 4.1.2, and Definition 1.2.4(ii), this thesis finds that, for any $m \in \mathbb{N}$, $\{x_j + B_{l_j}\}_{j=1}^m \subset \mathcal{B}$ with both $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ and $\{l_j\}_{j=1}^m \subset \mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m \subset (0, \infty)$,

$$
\begin{aligned}
&\left\|\left\{\sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \\
&\quad \times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} \left|\phi_k * h_j^{(2)}(x)\right|^2 \, dx\right]^{\frac{1}{2}} \\
&\lesssim J_1 \left\|\left\{\sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+\tau}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+\tau}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \\
&\quad \times \sum_{j=1}^m J_2^{(j)} \left\{\left[\int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j+\tau}} h(x)\right|^2 \, dx\right]^{\frac{1}{2}}\right. \\
&\quad \left. + \frac{1}{|x_j + B_{l_j}|^{\frac{1}{2}}} \int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j+\tau}} h(x)\right| \, dx\right\} \\
&\lesssim \left\|\left\{\sum_{i=1}^m \left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i+\tau}}\|_X}\right)^{\theta_0} \mathbf{1}_{x_i+B_{l_i+\tau}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1} \\
&\quad \times \sum_{j=1}^m \frac{\lambda_j |x_j + B_{l_j+\tau}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j+\tau}}\|_X} \left\{\left[\int_{x_j+B_{l_j+\tau}} \left|h(x) - P^d_{x_j+B_{l_j+\tau}} h(x)\right|^2 \, dx\right]^{\frac{1}{2}}\right.
\end{aligned}
$$

$$+\ \frac{1}{|x_j+B_{l_j}|^{\frac12}}\int_{x_j+B_{l_{j+\tau}}}\left|h(x)-P^d_{x_j+B_{l_{j+\tau}}}h(x)\right|\,dx\Bigg\}$$
$$\le \|h\|_{\mathcal{L}^A_{X,2,d,\theta_0}(\mathbb{R}^n)}+\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)},$$

where

$$J_1:=\frac{\|\{\sum_{i=1}^m(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_{i+\tau}}}\|_X})^{\theta_0}\mathbf{1}_{x_i+B_{l_{i+\tau}}}\}^{\frac{1}{\theta_0}}\|_X}{\|\{\sum_{i=1}^m(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X})^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\}^{\frac{1}{\theta_0}}\|_X}$$

and, for any $j\in\{1,\dots,m\}$,

$$J_2^{(j)}:=\frac{\|\mathbf{1}_{x_j+B_{l_{j+\tau}}}\|_X}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\frac{\lambda_j|x_j+B_{l_{j+\tau}}|^{\frac12}}{\|\mathbf{1}_{x_j+B_{l_{j+\tau}}}\|_X}.$$

This, combined with $p_0\in(\theta_0,2)$ and Corollary 2.2.7, further implies that

s5e6 (4.2.9)
$$\left\|\left\{\sum_{i=1}^m\left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1}\sum_{j=1}^m\frac{\lambda_j|x_j+B_{l_j}|^{\frac12}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}$$
$$\times\left[\sum_{k\in\mathbb{Z}}\int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}}\left|\phi_k*h_j^{(2)}(x)\right|^2dx\right]^{\frac12}$$
$$\lesssim\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.$$

Finally, this thesis deal with $h_j^{(3)}$. To do this, letting $s\in(0,\theta_0)$ and $\varepsilon\in(\frac{\ln b}{\ln(\lambda_-)}[\frac2s+d\frac{\ln(\lambda_+)}{\ln b}],\infty)$, this thesis has, for any $j\in\{1,\dots,m\}$ and $(x,k)\in\widehat{x_j+B_{l_j}}$,

$$\begin{aligned}
\left|\phi_k*h_j^{(3)}(x)\right|&\lesssim\int_{(x_j+B_{l_{j+\tau}})^\complement}\frac{b^{\varepsilon k\frac{\ln\lambda_-}{\ln b}}}{[b^k+\rho(x-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}}\left|h(y)-P^d_{x_j+B_{l_j}}h(y)\right|dy\\
&\sim\int_{(x_j+B_{l_{j+\tau}})^\complement}\frac{b^{\varepsilon k\frac{\ln\lambda_-}{\ln b}}}{[b^k+\rho(x_j-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}}\left|h(y)-P^d_{x_j+B_{l_j}}h(y)\right|dy\\
&\le\frac{b^{\varepsilon k\frac{\ln\lambda_-}{\ln b}}}{b^{\varepsilon l_j\frac{\ln\lambda_-}{\ln b}}}\int_{(x_j+B_{l_{j+\tau}})^\complement}\frac{b^{\varepsilon l_j\frac{\ln\lambda_-}{\ln b}}}{[\rho(x_j-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}}\left|h(y)-P^d_{x_j+B_{l_j}}h(y)\right|dy\\
&\lesssim\frac{b^{\varepsilon k\frac{\ln\lambda_-}{\ln b}}}{b^{\varepsilon l_j\frac{\ln\lambda_-}{\ln b}}}\int_{(x_j+B_{l_{j+\tau}})^\complement}\frac{b^{\varepsilon l_j\frac{\ln\lambda_-}{\ln b}}}{b^{l_j(1+\varepsilon\frac{\ln\lambda_-}{\ln b})}+[\rho(x_j-y)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}}\\
&\quad\times\left|h(y)-P^d_{x_j+B_{l_j}}h(y)\right|dy.
\end{aligned}$$

From this and Theorem 4.1.1, it follows that, for any $m\in\mathbb{N}$, $\{x_j+B_{l_j}\}_{j=1}^m\subset\mathcal{B}$ with both $\{x_j\}_{j=1}^m\subset\mathbb{R}^n$ and $\{l_j\}_{j=1}^m\subset\mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$,

$$\begin{aligned}&\left\|\left\{\sum_{i=1}^m\left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1}\sum_{j=1}^m\frac{\lambda_j|x_j+B_{l_j}|^{\frac12}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\\&\quad\times\left[\sum_{k\in\mathbb{Z}}\int_{\{x\in\mathbb{R}^n:(x,k)\in\widehat{x_j+B_{l_j}}\}}\left|\phi_k*h_j^{(3)}(x)\right|^2\,dx\right]^{\frac12}\\&\quad\lesssim\left\|\left\{\sum_{i=1}^m\left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1}\sum_{j=1}^m\frac{\lambda_j|x_j+B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\\&\quad\quad\times\sum_{k=-\infty}^{l_j}b^{-(l_j-k)\varepsilon\frac{\ln\lambda_-}{\ln b}}\int_{(x_j+B_{l_j+\tau})^\complement}\frac{b^{\varepsilon l_j\frac{\ln\lambda_-}{\ln b}}|h(x)-P^d_{x_j+B_{l_j}}h(x)|}{b^{l_j(1+\varepsilon\frac{\ln\lambda_-}{\ln b})}+[\rho(x_j-x)]^{1+\varepsilon\frac{\ln\lambda_-}{\ln b}}}\,dx\\&\quad\lesssim\|h\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}\sim\|h\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)}.\end{aligned}$$

Combining this, (4.2.6), (4.2.7), and (4.2.9), we conclude that

$$\begin{aligned}&\left\|\left\{\sum_{i=1}^m\left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1}\sum_{j=1}^m\frac{\lambda_j|x_j+B_{l_j}|^{\frac12}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\\&\quad\times\left[\sum_{k\in\mathbb{Z}}\int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}}|\phi_k*h(x)|^2\,dx\right]^{\frac12}\\&\quad\lesssim\|h\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)},\end{aligned}$$

which, together with the arbitrariness of $m\in\mathbb{N}$, $\{x_j+B_{l_j}\}_{j=1}^m\subset\mathcal{B}$ with both $\{x_j\}_{j=1}^m\subset\mathbb{R}^n$ and $\{l_j\}_{j=1}^m\subset\mathbb{Z}$, and $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$, further implies that, for any $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$,

$$d\mu(x,k):=|\phi_k*h(x)|^2\,dx$$

is an X–Carleson measure on $\mathbb{R}^n\times\mathbb{Z}$. Moreover, there exists a positive constant C, independent of b, such that $\|d\mu\|_X^{A,\theta_0}\lesssim\|h\|_{\mathcal{L}^{A}_{X,1,d,\theta_0}(\mathbb{R}^n)}$. This finishes the proof of (i).

This thesis now prove (ii). To this end, let $f\in H^{A,\infty,d,\theta_0}_{X,\mathrm{fin}}(\mathbb{R}^n)$ with the quasi-norm greater than zero. Then $f\in L^\infty(\mathbb{R}^n)$ with compact support. From this, the assumption that $h\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, and [33, (2.10)], it follows that

s5e7 (4.2.10) $$\left|\int_{\mathbb{R}^n}f(x)\overline{h(x)}\,dx\right|\sim\left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\phi_k*f(x)\overline{\phi_k*h(x)}\,dx\right|.$$

In addition, by the assumption that $f \in H_X^A(\mathbb{R}^n)$ and Theorem 3.1.4, this thesis finds that

$$\|\phi_k * f\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})} \sim \|f\|_{H_X^A(\mathbb{R}^n)} < \infty,$$

which, combined with Lemma 4.2.7, further implies that there exists a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of anisotropic (T_X^A,∞)-atoms supported, respectively, in $\{\widehat{x_j + B_{l_j}}\}_{j\in\mathbb{N}}$ with $\{x_j + B_{l_j}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ such that, for almost every $(x,k) \in \mathbb{R}^n \times \mathbb{Z}$,

$$\phi_k * f(x) = \sum_{j\in\mathbb{N}} \lambda_j A_j(x,k)$$

and

$$0 < \left\| \left\{ \sum_{j\in\mathbb{N}} \left(\frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \right)^{\theta_0} \mathbf{1}_{x_j+B_{l_j}} \right\}^{\frac{1}{\theta_0}} \right\|_X \lesssim \|f\|_{H_X^A(\mathbb{R}^n)}.$$

From this, (4.2.10), the Hölder inequality, the size condition of A_j, and the Tonelli theorem, this thesis infers that, for any $f \in H_{X,\,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)\overline{h(x)}\,dx\right| \\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_j \int_{\mathbb{R}^n} |A_j(x,k)|\,|\phi_k * h(x)|\,dx \\
&\quad\le \sum_{j\in\mathbb{N}} \lambda_j \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |A_j(x,k)|^2\,dx\right]^{\frac{1}{2}} \\
&\qquad\times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2\,dx\right]^{\frac{1}{2}} \\
&\quad= \sum_{j\in\mathbb{N}} \lambda_j \left[\sum_{k\in\mathbb{Z}} b^{-k} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\Gamma(y)\}} |A_j(x,k)|^2\,dx \int_{\{y\in\mathbb{R}^n:\,y\in x+B_k\}} dy\right]^{\frac{1}{2}} \\
&\qquad\times \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2\,dx\right]^{\frac{1}{2}} \\
&\quad= \sum_{j\in\mathbb{N}} \lambda_j \|A_j\|_{T_2^{A,2}(\mathbb{R}^n\times\mathbb{Z})} \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2\,dx\right]^{\frac{1}{2}} \\
&\quad\le \sum_{j\in\mathbb{N}} \frac{\lambda_j |x_j+B_{l_j}|^{\frac{1}{2}}}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X} \left[\sum_{k\in\mathbb{Z}} \int_{\{x\in\mathbb{R}^n:\,(x,k)\in\widehat{x_j+B_{l_j}}\}} |\phi_k * h(x)|^2\,dx\right]^{\frac{1}{2}} \\
&\quad\lesssim \|f\|_{H_X^A(\mathbb{R}^n)} \widetilde{\|d\mu\|}_X^{A,\theta_0},
\end{aligned}$$

which, together with Theorem 2.2.6, Proposition 4.2.2, and Corollary 2.2.7, further implies that

$$\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)} \lesssim \|d\mu\|_X^{A,\theta_0}.$$

This finishes the proof of (ii) and hence Theorem 4.2.3. □

Chapter 5

Several Applications

c5

In this section, this thesis applies Theorems 2.2.6, 3.1.4, 3.1.5, 3.1.6, 3.2.1, 3.3.1, 3.3.2, 4.1.1, 4.1.3,and 4.2.3 as well as Corollary 2.2.7 to seven concrete examples of ball quasi-Banach function spaces, namely Morrey spaces (see section 5.1 below), Orlicz-slice spaces (see Subsection 5.2 below), Lorentz spaces (see section 5.3 below), variable Lebesgue spaces (see section 5.4 below), mixed-norm Lebesgue spaces (see Subsection 5.5 below), weighted Lebesgue spaces (see section 5.6 below), and Orlicz spaces (see section 5.7 below). Particularly, in section 5.1, we give an example to point out that Theorems 3.2.1, 3.3.1, and 3.3.2 can not be applied to the Morrey space because its norm lacks concavity.

5.1 Morrey Spaces

s6-appl1

Recall that the classical Morrey space $M_q^p(\mathbb{R}^n)$ with $0 < q \le p < \infty$, originally introduced by Morrey [78] in 1938, plays a fundamental role in harmonic analysis and partial differential equations. From then on, various variants of Morrey spaces over different underlying spaces have been investigated and developed (see, for instance, [18, 83]).

Definition 5.1.1. Let A be a dilation and $0 < q \le p < \infty$. The *anisotropic Morrey space* $M^p_{q,A}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{M^p_{q,A}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \left[|B|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B)}\right] < \infty,$$

where $\mathcal{B}$ is the same as in (1.2.3).

It is easy to show that, $M^p_{q,A}(\mathbb{R}^n)$ is a ball quasi-Banach function space. From this and [97, Remark 8.4], this thesis deduces that $M^p_{q,A}(\mathbb{R}^n)$ satisfies both Assumptions 1.2.10 and 1.2.12 with $X := M^p_{q,A}(\mathbb{R}^n)$, $p_- \in (0, q]$, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (p, \infty)$, where $\underline{p} := \min\{p_-, 1\}$. In what follows, this thesis always lets $HM^p_{q,A}(\mathbb{R}^n)$ be the *anisotropic Hardy–Morrey space* which is defined to be the same as in Definition 2.1.1

with $X := M^p_{q,A}(\mathbb{R}^n)$. Then, applying Theorems 3.1.4, 3.1.5, and 3.1.6, this thesis obtains the following characterizations of $HM^p_{q,A}(\mathbb{R}^n)$, respectively, in terms of the anisotropic Lusin area function, the anisotropic Littlewood–Paley g-function, and the anisotropic Littlewood–Paley g^*_λ-function.

Thsm **Theorem 5.1.2.** *Let A be a dilation and $0 < q \le p < \infty$. Then Theorems 3.1.4, 3.1.5, and 3.1.6 with $X := M^p_{q,A}(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1,q\},\infty)$ hold true.*

3.23.x1 **Remark 5.1.3.** (i) This thesis point out that Theorem 5.1.2 is completely new.

(ii) However, Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 can not be applied to the anisotropic Morrey space $M^p_{q,A}(\mathbb{R}^n)$ because $M^p_{q,A}(\mathbb{R}^n)$ does not have an absolutely continuous quasi-norm.

Moreover, $M^p_{q,A}(\mathbb{R}^n)$ may not be q_0-concave. Indeed, let $A := 2I_{n\times n}$, $0 < q < p < \infty$, and $q_0 \in (p, 1]$. Assume that $\{f_k\}_{k=1}^\infty$ are the same indicator functions of the cubes with volume 1 as in [41, (2.4)] with $q := \frac{q}{q_0}$, $p := \frac{p}{q_0}$, and $r = \infty$ therein. Then, by [41, Theorem 2.15] with the same Φ therein, this thesis concludes that

$$\left\|\sum_{k=1}^\infty f_k\right\|_{M^{\frac{p}{q_0}}_{\frac{q}{q_0}}(\mathbb{R}^n)} = \|\Phi(a)\|_{M^{\frac{p}{q_0}}_{\frac{q}{q_0}}(\mathbb{R}^n)} \sim \|a\|_{l^\infty} = 1 < \sum_{k=1}^\infty \|f_k\|_{M^{\frac{p}{q_0}}_{\frac{q}{q_0}}(\mathbb{R}^n)} = \infty,$$

where $a := (1, \dots) \in l^\infty$. Thus, $M^p_q(\mathbb{R}^n)$ is not q_0-concave. Therefore, Theorems 3.2.1, 3.3.1, and 3.3.2 can not be applied to Morrey spaces and have their limitation because of their dependence on the concavity of norms.

However, when $A := 2I_{n\times n}$, $0 < q \le p \le 1$, and $X := M^p_{q,A}(\mathbb{R}^n)$, Theorems 3.2.1, 3.3.1, and 3.3.2 were obtained by de Almeida and Tiago, respectively, in [28, Theorem 3.3, Remark 3.4, and Proposition 3.8] via using a quite different atomic characterization of Hardy–Morrey spaces from Lemma 3.2.4 to avoid the dependence on the concavity of $\|\cdot\|_{M^p_q(\mathbb{R}^n)}$. But, for a general dilation A, this is still unclear so far.

s6-appl2 5.2 Orlicz-Slice Spaces

Recently, Zhang et al. [106] originally introduced the Orlicz-slice space on $\mathbb{R}^n$, which generalizes both the slice space in [2] and the Wiener-amalgam space in [29]. They also introduced the Orlicz-slice (local) Hardy spaces and developed a complete real-variable theory of these spaces in [105, 106]. For more studies about Orlicz-slice spaces, this thesis refers the reader to [44, 45].

Recall that a function $\Phi : [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is non-decreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0,\infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$. The function Φ is said to be of *upper* (resp. *lower*) *type* p for some $p \in [0,\infty)$ if there exists a positive constant C such that, for any $s \in [1,\infty)$ (resp. $s \in [0,1]$) and $t \in [0,\infty)$, $\Phi(st) \le Cs^p\Phi(t)$. The *Orlicz space* $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)\, dx \le 1\right\} < \infty.$$

Definition 5.2.1. Let A be a dilation, $\ell \in \mathbb{Z}$, $q \in (0,\infty)$, and Φ be an Orlicz function. The *anisotropic Orlicz-slice space* $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}} < \infty,$$

where B_ℓ is the same as in (1.2.2).

Let A be a dilation, $\ell \in \mathbb{Z}$, $q \in (0,\infty)$, and Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. Then, by the arguments similar to those used in the proofs of [106, Lemmas 2.28 and 4.5], this thesis finds that $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ is a ball quasi-Banach function space and has an absolutely continuous quasi-norm. From these and [97, Remark 8.14], this thesis deduces that $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ satisfies both Assumptions 1.2.10 and 1.2.12 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$, $p_- \in (0, \min\{p_\Phi^-, q\}]$, $\theta_0 \in (0, \underline{p})$, and $p_0 \in (\max\{p_\Phi^+, q\}, \infty)$, where $\underline{p} := \min\{p_-, 1\}$. In what follows, this thesis always lets $(HE_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ denote the *anisotropic Orlicz-slice Hardy space* which is defined to be the same as in Definition 2.1.1 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$.

Moreover, by Theorems 2.2.6, 4.1.1, 3.1.4, 3.1.5, 3.1.6, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with X replaced by $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$, this thesis obtains the following conclusion.

rliczslice **Theorem 5.2.2.** *Let A be a dilation, $\ell \in \mathbb{Z}$, $q \in (0,\infty)$, and Φ be an Orlicz function with positive lower type p_Φ^-. Then*

(i) *Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ hold true;*

(ii) *Theorems 3.1.4, 3.1.5, and 3.1.6 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ and $\lambda \in (\frac{2}{\min\{1, p_\Phi^-, q\}}, \infty)$ also hold true.*

3.23.x2 **Remark 5.2.3.** This thesis point out that Theorem 5.2.2 is completely new.

Let $\ell \in \mathbb{Z}$, $q \in (0,1)$, Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+ satisfying $0 < p_\Phi^- \le p_\Phi^+ < 1$, and

$$N \in \mathbb{N} \cap \left[\left\lfloor\left(\frac{1}{\min\{p_\Phi^-, q\}} - 1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2, \infty\right).$$

Then, by [97, Remark 8.14], this thesis concludes that $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 2.1.1 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$, $p_- \in (0, \min\{p_\Phi^-, q\}]$, $\theta_0 \in (0, p_-)$, and $p_0 \in (\max\{p_\Phi^+, q\}, \infty)$. Moreover, choose $q_0 = 1$. On the one hand, from [106, Lemma 5.4], this thesis infer that, for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$,

$$\sum_{k=1}^\infty \|f_k\|_{[(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)]^{\frac{1}{q_0}}} \le \left\|\sum_{k=1}^\infty f_k\right\|_{[(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)]^{\frac{1}{q_0}}}.$$

On the other hand, this thesis has, for any $B \in \mathcal{B}$,

eqorliczs
$$\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} \gtrsim \min\left\{|B|, |B|^{\frac{1}{\theta_0}}\right\}. \tag{5.2.1}$$

Indeed, for any $B \in \mathcal{B}$ with $|B| \ge |B_\ell|$,

eqos1
$$\begin{aligned}\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} &= \left\{\int_{\mathbb{R}^n}\left[\frac{\|\mathbf{1}_B\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}} \\ &\gtrsim \left(\int_B 1\,dx\right)^{\frac{1}{q}} = |B|^{\frac{1}{q}}.\end{aligned} \tag{5.2.2}$$

On the other hand, for any $x_0 \in \mathbb{R}^n$, $k \in \mathbb{Z}$ with $|B_k| \le |B_\ell|$, $x \in B(x_0, \lambda_-^\ell)$, and $\eta \in (0, p_\Phi^-)$, by [99, Remark 4.21(iv)], this thesis concludes that $L^\Phi(\mathbb{R}^n)$ satisfies Assumption 1.2.10 with $X := L^\Phi(\mathbb{R}^n)$, $u := 1/\eta$, and $p := \eta$. Thus, this thesis obtains

6.16.1
$$\|\mathbf{1}_{x_0+B_k}\|_{L^\Phi(\mathbb{R}^n)}^\eta \gtrsim \|[\mathcal{M}(\mathbf{1}_{x_0+B_k})]^{1/\eta}\|_{L^\Phi(\mathbb{R}^n)}^\eta. \tag{5.2.3}$$

For any $y \in x + B_\ell$, this thesis has

6.16.2
$$\mathcal{M}(\mathbf{1}_{x_0+B_k})(y) \ge \frac{1}{|B_\ell|}\int_{x+B_\ell} \mathbf{1}_{x_0+B_k}(z)\,dz \gtrsim \frac{|B_k|}{|B_\ell|}. \tag{5.2.4}$$

Combining (5.2.3) and (5.2.4), this thesis concludes that

$$\|\mathbf{1}_{x_0+B_k}\|_{L^\Phi(\mathbb{R}^n)}^\eta \gtrsim \left\|\left[\frac{|B_k|}{|B_\ell|}(\mathbf{1}_{x+B_\ell})\right]^{1/\eta}\right\|_{L^\Phi(\mathbb{R}^n)}^\eta \gtrsim |B_k|\,\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}^\eta.$$

Therefore, this thesis obtains

eqos2
$$\begin{aligned}\|\mathbf{1}_{x_0+B_k}\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} &= \left\{\int_{\mathbb{R}^n}\left[\frac{\|\mathbf{1}_{x_0+B_k}\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}} \\ &\gtrsim \left\{\int_{B(x_0,\lambda_-^\ell)}\left[\frac{\|\mathbf{1}_{x_0+B_k}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}}\end{aligned} \tag{5.2.5}$$

$$\gtrsim |B_k|^{1/\eta} \left\{ \int_{B(x_0,\lambda_-^\ell)} 1\, dx \right\}^{\frac{1}{q}} \sim |B_k|^{1/\eta}.$$

By (5.2.2) and (5.2.5), this thesis finds that, for any $B \in \mathcal{B}$,

$$\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} \gtrsim |B|^{1/q} \gtrsim |B| \text{ if } |B| \geq |B_\ell|$$

and

$$\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} \gtrsim |B|^{1/p_\Phi^-} \gtrsim |B|^{1/\theta_0} \text{ if } |B| \leq |B_\ell|.$$

This finishes the proof of (5.2.1).

Therefore, all the assumptions of Theorems 3.2.1, 3.3.1, and 3.3.2 with $X := (E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ are satisfied. Applying Theorems 3.2.1, 3.3.1, and 3.3.2, this thesis obtains the following conclusion.

liczslice2 **Theorem 5.2.4.** *Let $\ell \in \mathbb{Z}$, $q \in (0,1)$, and Φ be an Orlicz function with lower type p_Φ^- and upper type p_Φ^+ satisfying $0 < p_\Phi^- \leq p_\Phi^+ < 1$. Then Theorems 3.2.1, 3.3.1, and 3.3.2 with X replaced by $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ hold.*

liczslice2 **Remark 5.2.5.** This thesis point out that Theorem 5.2.4 even when $A := 2I_{n\times n}$ is completely new.

s6-appl3 5.3 Lorentz Spaces

Let $p \in (0,\infty]$ and $q \in (0,\infty]$. Recall that the *Lorentz space* $L^{p,q}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ with the following finite quasi-norm

6.13.x1 (5.3.1)
$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \left[\frac{q}{p}\int_0^\infty \left\{t^{\frac{1}{p}} f^*(t)\right\}^q \frac{dt}{t}\right]^{\frac{1}{q}} & \text{if } q \in (0,\infty), \\ \sup\limits_{t\in(0,\infty)} \left[t^{\frac{1}{p}} f^*(t)\right] & \text{if } q = \infty \end{cases}$$

with the usual modification made when $p = \infty$, where f^* denotes the *non-increasing rearrangement* of f, that is, for any $t \in (0,\infty)$,

$$f^*(t) := \{\alpha \in (0,\infty) : \ d_f(\alpha) \leq t\}$$

with $d_f(\alpha) := |\{x \in \mathbb{R}^n : \ |f(x)| > \alpha\}|$ for any $\alpha \in (0,\infty)$.

Then, by [99, Remarks 2.7(ii), 4.21(ii), and 6.8(iv)], this thesis concludes that $L^{p,q}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 2.1.1 with $X := L^{p,q}(\mathbb{R}^n)$, $p_- \in (0,p]$, $\theta_0 \in (0,\underline{p})$, and $p_0 \in (p,\infty)$, where $\underline{p} := \min\{p_-,1\}$, and has an absolutely continuous quasi-norm. In what follows, this thesis always lets $H_A^{p,q}(\mathbb{R}^n)$ be the *anisotropic Hardy–Lorentz space* which is defined to be the same as in Definition 2.1.1 with $X := L^{p,q}(\mathbb{R}^n)$. By Theorems 2.2.6, 3.1.4, 3.1.5, 3.1.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with X replaced by $L^{p,q}(\mathbb{R}^n)$, this thesis obtains the following conclusion.

thlorentz **Theorem 5.3.1.** *Let A be a dilation, $p \in (0,\infty)$, and $q \in (0,\infty]$. Then*

(i) *Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with $X := L^{p,q}(\mathbb{R}^n)$ hold true;*

(ii) *Theorems 3.1.4, 3.1.5, and 3.1.6 with $X := L^{p,q}(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1,p\},\infty)$ also hold true.*

Denote by $\mathcal{P}(\mathbb{R}^n)$ the *set of all the measurable functions* $p(\cdot)$ on $\mathbb{R}^n$ satisfying

s6e1 (5.3.2)
$$0 < \widetilde{p_-} := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: \widetilde{p_+} < \infty. \tag{5.3.2}$$

Denote by $C^{\log}(\mathbb{R}^n)$ the set of all the functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the *globally log-Hölder continuous condition*, that is, there exist $C_{\log}(p), C_\infty \in (0,\infty)$ and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log\left(e + 1/|x-y|\right)}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + \rho(x))}.$$

3.23.x3 **Remark 5.3.2.** Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in (0,\infty)$. This thesis point out that Theorem 5.3.1(i) is a special case of [71, Theorems 1 and 2] with $p(\cdot) \equiv p \in (0,\infty)$ therein and that Theorem 5.3.1(ii) improves the corresponding results in [76, Theorems 2.7, 2.8, and 2.9] by widening the range of $p \in (0,1]$ into $p \in (0,\infty)$. Although the variable Hardy–Lorentz space $L^{p(\cdot),q}(\mathbb{R}^n)$ is also a ball quasi-Banach function space, [71, Theorems 1 and 2] can not be deduced from Theorems 2.2.6 and 4.2.3. This is because the boundedness of the powered Hardy–Littlewood maximal operator on the associate space of $L^{p(\cdot),q}(\mathbb{R}^n)$ is still unknown, which makes it impossible to verify Assumption 1.2.12 with $X := L^{p(\cdot),q}(\mathbb{R}^n)$.

Moreover, choose $q_0 \in (\max\{p,q\},1]$. From [55, Theorem 6(iii)] and [40, Proposition 1.4.5(13)], this thesis deduces that there exists a positive constant C such that, for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$,

$$\begin{aligned}\left[\sum_{k=1}^\infty \|f_k\|_{L^{\frac{p}{q_0},\frac{q}{q_0}}(\mathbb{R}^n)}\right]^{\frac{q}{q_0}} &= \left[\sum_{k=1}^\infty \|f_k\|_{\Lambda_{\frac{q}{q_0},\frac{q}{p}t^{\frac{q}{p}-1}}}\right]^{\frac{q}{q_0}} = \left[\sum_{k=1}^\infty \left\|f_k^{\frac{q}{q_0}}\right\|_{\Lambda_{1,\frac{q}{p}t^{\frac{q}{p}-1}}}^{\frac{q_0}{q}}\right]^{\frac{q}{q_0}}\\ &\le C\left\|\left(\sum_{k=1}^\infty f_k\right)^{\frac{q}{q_0}}\right\|_{\Lambda_{1,\frac{q}{p}t^{\frac{q}{p}-1}}} = C\left\|\sum_{k=1}^\infty f_k\right\|_{\Lambda_{\frac{q}{q_0},\frac{q}{p}t^{\frac{q}{p}-1}}}^{\frac{q}{q_0}}\end{aligned}$$

$$= C \left\| \sum_{k=1}^{\infty} f_k \right\|_{L^{\frac{p}{q_0}, \frac{q}{q_0}}(\mathbb{R}^n)}^{\frac{q}{q_0}}$$

with Λ the same as in lines 6-7 of [55, p. 270], which further implies that

$$\sum_{k=1}^{\infty} \|f_k\|_{L^{\frac{p}{q_0}, \frac{q}{q_0}}(\mathbb{R}^n)} \le C \left\| \sum_{k=1}^{\infty} f_k \right\|_{L^{\frac{p}{q_0}, \frac{q}{q_0}}(\mathbb{R}^n)}.$$

Moreover, by (5.3.1), for any $B \in \mathcal{B}$,

$$\|\mathbf{1}_B\|_{L^{p,q}(\mathbb{R}^n)} = \left\{ \frac{q}{p} \int_0^{|B|} t^{\frac{q}{p}-1} dt \right\}^{\frac{1}{q}} = |B|^{\frac{1}{p}} \ge \min\left\{ |B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}} \right\}.$$

Therefore, all the assumptions of Theorems 3.2.1, 3.3.1, and 3.3.2 are satisfied with $X := L^{p,q}(\mathbb{R}^n)$. Applying Theorems 3.2.1, 3.3.1, and 3.3.2, this thesis obtains the following conclusion.

thlorentz2 **Theorem 5.3.3.** *If $p \in (0,1)$ and $q \in (0,1)$, then Theorems 3.2.1, 3.3.1, and 3.3.2 with X replaced by $L^{p,q}(\mathbb{R}^n)$ hold.*

remrentz2 **Remark 5.3.4.** This thesis point out that Theorem 5.3.3 even when $A := 2I_{n\times n}$ is completely new.

s6-appl4 5.4 Variable Lebesgue Spaces

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx < \infty,$$

equipped with the *quasi-norm* $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ defined by setting

6.8.x1 (5.4.1)
$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{ \lambda \in (0,\infty) : \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda}\right]^{p(x)} dx \le 1 \right\}.$$

Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then, by [99, Remarks 2.7(iv), 4.21(v), and 6.8(vii)], this thesis concludes that $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 2.1.1 with $X := L^{p(\cdot)}(\mathbb{R}^n)$, $p_- := \widetilde{p_-}$, $\theta_0 \in (0, \underline{\widetilde{p}})$, and $p_0 \in (\widetilde{p_+}, \infty]$, where $\widetilde{p_-}$ and $\widetilde{p_+}$ are the same as in (5.3.2) and $\underline{\widetilde{p}} := \min\{1, \widetilde{p_-}\}$, and has an absolutely continuous quasi-norm. In what follows, this thesis always lets $H_A^{p(\cdot)}(\mathbb{R}^n)$ be the *anisotropic variable Hardy space* which is defined to be the same as in Definition 2.1.1 with $X := L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, by Theorems 2.2.6, 3.1.4, 3.1.5, 3.1.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with X replaced by $L^{p(\cdot)}(\mathbb{R}^n)$, this thesis obtains the following conclusion.

thvariable **Theorem 5.4.1.** *Let A be a dilation and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then*

(i) *Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with $X := L^{p(\cdot)}(\mathbb{R}^n)$ hold true;*

(ii) *Theorems 3.1.4, 3.1.5, and 3.1.6 with $X := L^{p(\cdot)}(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, \widetilde{p}_-\}, \infty)$ also hold true, where $\widetilde{p_-}$ is the same as in* (5.3.2).

3.23.x4 **Remark 5.4.2.** This thesis point out that Theorem 5.4.1(i) was also obtained in [48, Theorems 1, 2, and 3, and Corollary 1] and Theorem 5.4.1(ii) improves the corresponding results in [72, Theorems 6.1, 6.2, and 6.3] by widening the range of $\lambda \in (1+2/\min\{2, \widetilde{p}_-\}, \infty)$ into $\lambda \in (2/\min\{1, \widetilde{p}_-\}, \infty)$.

Let A be a dilation and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $0 < \widetilde{p_-} \le \widetilde{p_+} < 1$ and

$$N \in \mathbb{N} \cap \left[\left\lfloor \left(\frac{1}{\widetilde{p_-}} - 1\right) \frac{\ln b}{\ln(\lambda_-)} \right\rfloor + 2, \infty\right).$$

Then, by [99, Remarks 2.7(iv) and 4.21(v)], this thesis concludes that $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies all assumptions of Definition 2.1.1 with $X := L^{p(\cdot)}(\mathbb{R}^n)$, $p_- \in (0, \widetilde{p_-}]$, $\theta_0 \in (0, p_-)$, and $p_0 \in (\widetilde{p_+}, \infty)$. Moreover, choose $q_0 \in (\widetilde{p_+}, 1]$. On the one hand, from [102, Remark 2.1(iv)], this thesis deduces that, for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$,

$$\sum_{k=1}^{\infty} \|f_k\|_{L^{\frac{p(\cdot)}{q_0}}(\mathbb{R}^n)} \le \left\|\sum_{k=1}^{\infty} f_k\right\|_{L^{\frac{p(\cdot)}{q_0}}(\mathbb{R}^n)}.$$

On the other hand, by (5.4.1), this thesis finds that, for any $B \in \mathcal{B}$,

$$\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \gtrsim \min\left\{|B|^{\frac{1}{\widetilde{p_+}}}, |B|^{\frac{1}{\widetilde{p_-}}}\right\} > \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\}.$$

Thus, $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies all the assumptions of Theorem 3.2.1 with $X := L^{p(\cdot)}(\mathbb{R}^n)$. In this case, Theorems 3.2.1, 3.3.1 and 3.3.2 were obtained, respectively, in [67, Theorems 1, 2 and 3].

s6-appl5 5.5 Mixed-Norm Lebesgue Spaces

Let $\vec{p} := (p_1, \ldots, p_n) \in (0, \infty]^n$. Recall that the *mixed-norm Lebesgue space* $L^{\vec{p}}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

6.8.x2 (5.5.1)
$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}} \cdots \left[\int_{\mathbb{R}} \left\{\int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1}\, dx_1\right\}^{\frac{p_2}{p_1}} dx_2\right]^{\frac{p_3}{p_2}} \cdots dx_n\right\}^{\frac{1}{p_n}}$$

is finite with the usual modifications made when $p_i=\infty$ for some $i\in\{1,\ldots,n\}$.

Let $\vec{p}\in(0,\infty]^n$. Then, by both [104, p. 2047] and [47, Lemmas 3.2 and 4.4], this thesis concludes that $L^{\vec{p}}(\mathbb{R}^n)$ satisfies all the assumptions of Definition 2.1.1 with $X:=L^{\vec{p}}(\mathbb{R}^n)$, $p_-:=\widehat{p_-}$, $\theta_0\in(0,\underline{\widehat{p}})$, and $p_0\in(\theta_0,\infty)$, where $\widehat{p_-}:=\min\{p_1,...,p_n\}$ and $\underline{\widehat{p}}:=\min\{1,\widehat{p_-}\}$, and has an absolutely continuous quasi-norm. In what follows, this thesis always lets $H_A^{\vec{p}}(\mathbb{R}^n)$ be the *anisotropic mixed-norm Hardy space* which is defined to be the same as in Definition 2.1.1 with $X:=L^{\vec{p}}(\mathbb{R}^n)$. Moreover, by Theorems 2.2.6, 3.1.4, 3.1.5, 3.1.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with X replaced by $L^{\vec{p}}(\mathbb{R}^n)$, this thesis obtains the following conclusion.

thmix **Theorem 5.5.1.** *Let A be a dilation and $\vec{p}\in(0,\infty)^n$. Then*

(i) *Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with $X:=L^{\vec{p}}(\mathbb{R}^n)$ hold true;*

(ii) *Theorems 3.1.4, 3.1.5, and 3.1.6 with $X:=L^{\vec{p}}(\mathbb{R}^n)$ and $\lambda\in(2/\min\{1,\widehat{p_-}\},\infty)$ also hold true, where $\widehat{p_-}:=\min\{p_1,...,p_n\}$.*

3.23.x5 **Remark 5.5.2.** (i) This thesis point out that Theorem 5.5.1(i) was also obtained in [49, Theorems 3.4, 4.1, and 5.3, and Corollary 3.9] and Theorem 5.5.1(ii) improves the corresponding results in [47, Theorems 6.2, 6.3, and 6.4] by widening the range of $\lambda\in(1+2/\min\{2,\widehat{p_-}\},\infty)$ into $\lambda\in(2/\min\{1,\widehat{p_-}\},\infty)$.

(ii) Let $\vec{a}:=(a_1,\ldots,a_n)\in[1,\infty]^n$. Then Theorem 5.5.1(i) with

$$A:=\begin{pmatrix}2^{a_1} & 0 & \cdots & 0\\ 0 & 2^{a_2} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 2^{a_n}\end{pmatrix}$$

gives the dual space of the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ which was introduced in [20, Definition 3.3] and completely answers the open problem on the dual space of $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ proposed in [20].

Let $\vec{p}\in(0,1)^n$, choose $q_0\in(\widehat{p_+},1]$. From (5.5.1) and [68, (9)], this thesis deduces that, for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$ and any $B\in\mathcal{B}$,

$$\sum_{k=1}^\infty\|f_k\|_{L^{\frac{\vec{p}}{q_0}}(\mathbb{R}^n)}\le C\left\|\sum_{k=1}^\infty f_k\right\|_{L^{\frac{\vec{p}}{q_0}}(\mathbb{R}^n)}$$

and

$$\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)} \gtrsim \min\left\{|B|^{\frac{1}{\widehat{p_+}}}, |B|^{\frac{1}{\widehat{p_-}}}\right\} = \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\}.$$

Thus, $L^{\vec{p}}(\mathbb{R}^n)$ satisfies all the assumptions of Theorem 3.2.1 with $X := L^{\vec{p}}(\mathbb{R}^n)$. In this case, Theorems 3.2.1, 3.3.1 and 3.3.2 were obtained, respectively, in [68, Theorems 3.1, 4.1 and 4.3].

s6-appl6 5.6 Weighted Lebesgue Spaces

Let $p \in (0, \infty]$ and $w \in \mathcal{A}_\infty(A)$. From [99, Remarks 2.7(iii), 4.21(iii), and 6.8(v)], this thesis deduces that $L^p_w(\mathbb{R}^n)$ satisfies all the assumptions of Definition 2.1.1 with $X := L^p_w(\mathbb{R}^n)$, $p_- \in (0, p/q_w]$, $\theta_0 \in (0, \min\{1, p_-\})$, and $p \in (\theta_0, \infty)$, where q_w is the same as in (3.1.10), and has an absolutely continuous quasi-norm. In what follows, this thesis always lets $H^p_w(\mathbb{R}^n)$ be the *anisotropic weighted Hardy space* which is defined to be the same as in Definition 2.1.1 with $X := L^p_w(\mathbb{R}^n)$. By Theorems 2.2.6, 3.1.4, 3.1.5, 3.1.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with X replaced by $L^p_w(\mathbb{R}^n)$, this thesis obtains the following conclusion.

thwei **Theorem 5.6.1.** *Let A be a dilation, $p \in (0, \infty)$, and $w \in \mathcal{A}_\infty(A)$. Then*

(i) *Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with $X := L^p_w(\mathbb{R}^n)$ hold true;*

(ii) *Theorems 3.1.4, 3.1.5, and 3.1.6 with $X := L^p_w(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, q_w/p\}, \infty)$ also hold true, where q_w is the same as in* (3.1.10).

3.27.x1 **Remark 5.6.2.** This thesis point out that Theorem 5.6.1(i) is completely new and Theorem 5.6.1(ii) improves the corresponding results in [58, Theorems 2.14, 3.1, and 3.9] by widening the range of $p \in (0, 1]$ into $p \in (0, \infty)$. However, Theorems 3.2.1, 3.3.1, and 3.3.2 can not be applied to Weighted Lebesgue Spaces since (3.2.2) may not hold true when $X := L^p_w(\mathbb{R}^n)$.

s6-appl7 5.7 Orlicz Spaces

Let Φ be an Orlicz function with positive lower type p_Φ^- and positive upper type p_Φ^+. From [99, Remarks 2.7(iii), 4.21(iv), and 6.8(vi)], this thesis deduces that $L^\Phi(\mathbb{R}^n)$ satisfies all the assumptions of Definition 2.1.1 with $X := L^\Phi(\mathbb{R}^n)$, $p_- \in (0, p_\Phi^-]$, $\theta_0 \in (0, \min\{p_\Phi^-, 1\})$, and $p_0 \in (\max\{p_\Phi^+, 1\}, \infty)$, and has an absolutely continuous quasi-norm. In what follows, this thesis always lets $H^\Phi_A(\mathbb{R}^n)$ be the *anisotropic Orlicz–Hardy space* which is defined to be the same as in Definition 2.1.1 with $X := L^\Phi(\mathbb{R}^n)$. Moreover, by Theorems 2.2.6, 3.1.4, 3.1.5, 3.1.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with X replaced by $L^\Phi(\mathbb{R}^n)$, this thesis obtains the following conclusion.

thorlicz **Theorem 5.7.1.** *Let A be a dilation and Φ an Orlicz function with lower type $p_\Phi^- \in (0,\infty)$. Then*

(i) *Theorems 2.2.6, 4.1.1, 4.1.3, and 4.2.3 as well as Corollary 2.2.7 with $X := L^\Phi(\mathbb{R}^n)$ hold true;*

(ii) *Theorems 3.1.4, 3.1.5, and 3.1.6 with $X := L^\Phi(\mathbb{R}^n)$ and $\lambda \in (2/\min\{1, p_\Phi^-\}, \infty)$ also hold true.*

3.27.x2 **Remark 5.7.2.** This thesis point out that Theorem 5.7.1(i) is completely new and Theorem 5.7.1(ii) improves the corresponding results in [58, Theorems 2.14, 3.1, and 3.9] by widening the range of $p_\Phi^- \in (0,1]$ into $p_\Phi^- \in (0,\infty)$.

Moreover, choose $q_0 \in (p_\Phi^+, 1]$. Then, from [106, Remark 5.3] and [46, (25)], this thesis deduces that, for any non-negative measurable functions $\{f_k\}_{k=1}^\infty$ and any $B \in \mathcal{B}$,

$$\sum_{k=1}^\infty \|f_k\|_{[L^\Phi(\mathbb{R}^n)]^{\frac{1}{q_0}}} \le \left\| \sum_{k=1}^\infty f_k \right\|_{[L^\Phi(\mathbb{R}^n)]^{\frac{1}{q_0}}}$$

and

$$\|\mathbf{1}_B\|_{L^\Phi(\mathbb{R}^n)} \gtrsim \min\left\{ |B|^{\frac{1}{p_\Phi^-}}, |B|^{\frac{1}{p_\Phi^+}} \right\} \ge \min\left\{ |B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}} \right\}.$$

Therefore, all the assumptions of Theorems 3.2.1, 3.3.1, and 3.3.2 are satisfied with $X := L^\Phi(\mathbb{R}^n)$. Applying Theorems 3.2.1, 3.3.1, and 3.3.2, this thesis obtains the following conclusion.

thorlicz2 **Theorem 5.7.3.** *Let Φ be an Orlicz function with lower type p_Φ^- and upper type p_Φ^+ satisfying $0 < p_\Phi^- \le p_\Phi^+ < 1$. Then Theorems 3.2.1, 3.3.1, and 3.3.2 with X replaced by $L^\Phi(\mathbb{R}^n)$ hold.*

rerlicz2 **Remark 5.7.4.** This thesis point out that Theorem 5.7.3 even when $A := 2I_{n\times n}$ is completely new.

中文详细摘要

在数学和物理学等领域中, 函数空间及其实变特征一直是现代调和分析的研究重点之一. 许多经典函数空间正是在解决这些问题的过程中被引入和发展的. 在欧氏空间 $\mathbb{R}^n$ 上, 经典 Hardy 空间的对偶理论在分析学的许多分支中发挥着重要作用, 迄今为止已经得到了系统的考虑和发展; 在实 Hardy 空间上, 有著名的对偶定理, 即有界平均振荡函数空间 $\mathrm{BMO}(\mathbb{R}^n)$, 是 Hardy 空间 $H^1(\mathbb{R}^n)$ 的对偶空间, 这是由 Fefferman 和 Stein[34] 提出的. 此外, 值得指出的是, Taibleson 和 Weiss[88] 给出了 $p \in (0,1]$ 的 Hardy 空间 $H^p(\mathbb{R}^n)$ 的完整对偶理论, 在该理论中,$H^p(\mathbb{R}^n)$ 的对偶空间被证明是由 Campanato[16] 引入的特殊 Campanato 空间.

除此之外, 1972 年, Fefferman 和 Stein[34] 提出了一个著名的问题, 即如何刻画经典 Hardy 空间 $H^p(\mathbb{R}^n)$ 中的函数 f 的 Fourier 变换 $\widehat{f}$. 1980 年, Taibleson 和 Weiss[88] 证明了, 对于任意给定的 $p \in (0,1]$,$f \in H^p(\mathbb{R}^n)$ 的 Fourier 变换与连续函数 F 在分布意义下一致, 并由此得到了 Hardy–Littlewood 不等式的推广 (参见 [86, p. 128]).

最近, Sawano 等人 [81] 首次引入了球拟 Banach 函数空间 X, 进一步推广了 [3] 中的 Banach 函数空间, 还引入了与 X 相关的 Hardy 空间 $H_X(\mathbb{R}^n)$, 这为 Hardy 型空间的研究提供了一个框架. 另一方面, 从 1970 年代开始, 人们对将调和分析中出现的经典函数空间从 $\mathbb{R}^n$ 扩展到各种各样的各向异性设置和其他域中的兴趣日益增长, 参见, [22, 36, 38, 39, 42, 77, 85, 89, 91, 92]. 2003 年, Bownik[4] 引入并研究了各向异性 Hardy 空间 $H^p_A(\mathbb{R}^n)$, 其中 $p \in (0,\infty)$,A 是 $\mathbb{R}^n$ 上的一般扩张矩阵. Wang 等人 [97] 首次引入并研究的与 A 和 X 都相关的各向异性 Hardy 空间 $H^A_X(\mathbb{R}^n)$, 其中他们通过极大函数、原子、有限原子和分子刻画 $H^A_X(\mathbb{R}^n)$, 并得到了 $H^A_X(\mathbb{R}^n)$ 上的各向异性 Calderón–Zygmund 算子的有界性. 受此和 [103] 启发, 一个非常**自然的问题**出现了: 能否建立 $H^A_X(\mathbb{R}^n)$ 上的各向异性 Littlewood–Paley 函数特征及 Fourier 变换, 并证明 $H^A_X(\mathbb{R}^n)$ 的对偶空间是否为各向异性球 Campanato 函数空间, 通过 Carleson 测度来刻画这个空间?

本文对这个问题给出肯定的回答, 并丰富各向异性球 Campanato 空间以及与 A 和 X 都相关的各向异性球 Hardy 空间的实变特征. 本文通过将原子的有限线性组合作为一个整体来考虑, 引入了各向异性球 Campanato 函数空间. 利用该空间以及两个温和的假设, 摆脱了对 $\|\cdot\|_X$ 的凹性依赖, 证明了 $H^A_X(\mathbb{R}^n)$ 的对偶空间正是各向异性球 Campanato 函数空间. 此外, 通过将 X 嵌入到某个各向异性加权 Lebesgue 空间中, 本文建立了 $H^A_X(\mathbb{R}^n)$ 的各向异性 Littlewood–Paley 函数特征. 本文还证明了 $f \in H^A_X(\mathbb{R}^n)$ 的 Fourier 变换 $\widehat{f}$ 与 $\mathbb{R}^n$ 上的一个连续函数 F 在广义分布的意义下一致, 应用这一点和关于原子的 Fourier 变换值的一个技术不等式, 进一步得到了 $H^A_X(\mathbb{R}^n)$ 中的 Hardy–Littlewood 不等式的推广. 结合 $H^A_X(\mathbb{R}^n)$ 的对偶定理和与 X 相关的各向异性帐篷空间的原子分解, 可以得到了各向异性球 Campanato 函数空间的 Carleson 测度特征. 最后本文将以上的所有结果应用于具体

的函数空间, 对经典结果进行了推广. 具体地, 本文主要研究了以下四个方面.

一、$H_X^A(\mathbb{R}^n)$ 和 $\mathcal{L}^A_{X,q',d,\theta_0}(\mathbb{R}^n)$ 的对偶性

在这一节中, 首先对符号做一些约定.

令 $\mathbb{N} := \{1, 2, \dots\}$,$\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$,$\mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, 以及 $\mathbf{0}$ 表示 $\mathbb{R}^n$ 的*原点*. 对于任意的多重指标 $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ 和任意的 $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, 定义 $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$, 以及 $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. 用 C 表示一个*正常数*, 它的取值所涉及的主要参数无关, 但可能会在不同的行取不同的值. 使用 $C_{(\alpha,\dots)}$ 表示依赖于指定参数 $\alpha, \dots$ 的正常数. 符号 $f \lesssim g$ 表示 $f \leq Cg$. 若 $f \lesssim g$ 且 $g \lesssim f$, 则写作 $f \sim g$. 若 $f \leq Cg$ 且 $g = h$ 或 $g \leq h$, 则写作 $f \lesssim g = h$ 或 $f \lesssim g \leq h$. 对于任意 $q \in [1, \infty]$, 用 q' 表示其*共轭指标*, 即, $1/q + 1/q' = 1$. 对于任意 $x \in \mathbb{R}^n$, 用 $|x|$ 表示 x 的 n *维欧几里得度量*. 若 E 是 $\mathbb{R}^n$ 的子集, 用 $\mathbf{1}_E$ 表示其*特征函数*, 用 $E^\complement$ 表示集合 $\mathbb{R}^n \setminus E$. 对于任意的 $r \in (0, \infty)$ 和 $x \in \mathbb{R}^n$, 用 $B(x, r)$ 表示以 x 为球心、半径为 r 的球, 即, $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. 对于任意的球 B, 用 x_B 表示其球心, r_B 表示其半径, 并且对于任意的 $\lambda \in (0, \infty)$, 用 λB 表示与 B 同心的球, 半径为 λr_B. 还使用 $\epsilon \to 0^+$ 表示存在 $\alpha_0 \in (0, \infty)$ 使得 $\epsilon \in (0, \alpha_0)$ 并且 $\epsilon \to 0$. 设 X 和 Y 是两个带有范数 $\|\cdot\|_X$ 和 $\|\cdot\|_Y$ 的完备向量空间; 用 $X \hookrightarrow Y$ 表示 $X \subset Y$ 并且存在正常数 C, 使得对于任意的 $f \in X$, $\|f\|_Y \leq C\|f\|_X$. 最后, 当证明一个定理 (或类似的结论) 时, 总是使用陈述该定理时所用的一致符号.

其次, 回顾一些关于伸缩 (参见, [4, 43]) 和球拟 Banach 函数空间 (参见, [60, 61, 81, 95, 96, 100, 104]) 的符号和概念. 首先回顾 [4] 中对*扩张矩阵*的概念.

h-dilation **定义 1.** 一个 $n \times n$ 的实矩阵 A 若满足

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

则称其为*扩张矩阵* (简称为*伸缩*), 其中 $\sigma(A)$ 表示 A 的所有特征值的集合.

设 A 是一个扩张矩阵, 并且

ch-2.14.x1 (1)
$$b := |\det A|,$$

其中 $\det A$ 表示 A 的*行列式*. 设 $A := (a_{i,j})_{1 \leq i,j \leq n}$ 是一个扩张矩阵, 则*矩阵范数*定义为

$$||A|| := (\sum_{i,j=1}^{n} |a_{i,j}|^2)^{1/2}.$$

然后由 [4, p. 6, (2.7)], 可以得出 $b \in (1, \infty)$, 并且存在一个开的对称椭球体 Δ, 使得 $|\Delta| = 1$ 成立, 以及一个 $r \in (1, \infty)$, 使得 $\Delta \subset r\Delta \subset A\Delta$ 成立 (参见 [4, p. 5, Lemma 2.2]), 可得,

对于任意 $k \in \mathbb{Z}$,

ch-B_k

$$B_k := A^k \Delta \tag{2}$$

都是开集, $B_k \subset rB_k \subset B_{k+1}$, 并且 $|B_k| = b^k$. 对于任意 $x \in \mathbb{R}^n$ 和 $k \in \mathbb{Z}$, 椭球 $x + B_k$ 被称为*伸缩球*. 在接下来的讨论中, 本文令 $\mathcal{B}$ 是所有这样的伸缩球的集合, 即,

ch-ball-B

$$\mathcal{B} := \{x + B_k : \ x \in \mathbb{R}^n, k \in \mathbb{Z}\} \tag{3}$$

并且令

ch-tau

$$\tau := \inf\left\{l \in \mathbb{Z} : \ r^l \geq 2\right\}. \tag{4}$$

设 $\lambda_-, \lambda_+ \in (0, \infty)$ 满足

ch-2.21.x1

$$1 < \lambda_- < \min\{|\lambda| : \ \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \ \lambda \in \sigma(A)\} < \lambda_+. \tag{5}$$

若 A 在 $\mathbb{R}$ 上是可对角化的, 则可以假设

$$\lambda_- := \min\{|\lambda| : \ \lambda \in \sigma(A)\} \text{ 和 } \lambda_+ := \max\{|\lambda| : \ \lambda \in \sigma(A)\}.$$

否则, 本文可以根据证明中所需, 将它们选得足够接近.

与伸缩 A 相关的齐次拟范数的定义如下 [4, p. 6, Definition 2.3].

ch-quasi-norm

定义 2. 与伸缩 A 相关的*齐次拟范数* 是一个可测映射 $\varrho : \ \mathbb{R}^n \to [0, \infty)$, 并满足以下条件:

(i) $\varrho(x) = 0 \Longleftrightarrow x = \mathbf{0}$, 其中 $\mathbf{0}$ 表示 $\mathbb{R}^n$ 的原点;

(ii) 对于任意 $x \in \mathbb{R}^n$ 有 $\varrho(Ax) = b\varrho(x)$;

(iii) 存在 $A_0 \in [1, \infty)$, 使得对于任意 $x, y \in \mathbb{R}^n$ 有

$$\varrho(x + y) \leq A_0 \left[\varrho(x) + \varrho(y)\right].$$

在经典欧式空间的情况下, 令 $A := 2\, I_{n\times n}$, 对于任意 $x \in \mathbb{R}^n$, 令 $\varrho(x) := |x|^n$, 此时 ϱ 是与 A 相关的齐次拟范数的一个例子. 在此处和后文中, $I_{n\times n}$ 总是表示 $n \times n$ 的*单位矩阵*, 其中 $|\cdot|$ 表示 $\mathbb{R}^n$ 中的*经典欧式空间范数*.

对于固定的伸缩 A, 由 [4, p. 6, Lemma 2.4], 本文定义以下的拟范数, 在后文中需要反复使用.

ch-def-shqn

定义 3. 定义在 $\mathbb{R}^n$ 上与伸缩 A 相关的*阶梯齐次拟范数* ρ 为,

$$\rho(x) := \begin{cases} b^k & \text{if } x \in B_{k+1} \backslash B_k, \\ 0 & \text{if } x = \mathbf{0}, \end{cases}$$

其中 b 与 (1) 一致, 而对于任意 $k \in \mathbb{Z}$, B_k 与 (2) 一致.

此时 $(\mathbb{R}^n, \rho, dx)$ 是 Coifman 和 Weiss [25] 意义下的齐型空间, 其中 dx 表示 n 维 Lebesgue 测度. 对于关于函数空间在齐型空间上的实变理论的更多研究, 可参见 [10, 11, 12, 62, 63, 64].

在本文中, A 总是定义 1 中是伸缩, b 与 (1) 一致, ρ 是在定义 3 中的阶梯齐次拟范数, $\mathcal{B}$ 是 (3) 中的所有伸缩球的集合, $\mathscr{M}(\mathbb{R}^n)$ 是 $\mathbb{R}^n$ 上所有可测函数的集合, 以及对于任意 $k \in \mathbb{Z}$, B_k 与 (2) 一致. 接下来, 回顾球拟 Banach 函数空间的定义 (参见 [81]).

ch-BQBFS **定义 4.** 赋拟范数的线性空间 $X \subset \mathscr{M}(\mathbb{R}^n)$, 其拟范数 $\|\cdot\|$ 对整个 $\mathscr{M}(\mathbb{R}^n)$ 有意义, 若它满足以下条件, 则称其为*球拟 Banach 函数空间*:

(i) 对于任意 $f \in \mathscr{M}(\mathbb{R}^n)$, $\|f\|_X = 0$ 可以得到 $f = 0$ 几乎处处成立;

(ii) 对于任意 $f, g \in \mathscr{M}(\mathbb{R}^n)$, 若几乎处处满足 $|g| \le |f|$, 则 $\|g\|_X \le \|f\|_X$;

(iii) 对于任意 $\{f_m\}_{m\in\mathbb{N}} \subset \mathscr{M}(\mathbb{R}^n)$ 和 $f \in \mathscr{M}(\mathbb{R}^n)$, 若 $m \to \infty$ 时, 几乎处处有 $0 \le f_m \uparrow f$, 则当 $m \to \infty$ 时, 几乎处处有 $\|f_m\|_X \uparrow \|f\|_X$;

(iv) 对于任意伸缩球 $B \in \mathcal{B}$, 有 $\mathbf{1}_B \in X$.

此外, 若球拟 Banach 函数空间 X 满足以下条件:

(v) 对于任意 $f, g \in X$ 有 $\|f+g\|_X \le \|f\|_X + \|g\|_X$;

(vi) 对于给定的伸缩球 $B \in \mathcal{B}$, 存在正常数 $C_{(B)}$, 使得对于任意 $f \in X$ 有

$$\int_B |f(x)|\, dx \le C_{(B)} \|f\|_X.$$

则称为*球 Banach 函数空间*.

ch-s1r1 **注记 5.** (i) 如 [97, Remark 2.5(i)] 中所述, 若 $f \in \mathscr{M}(\mathbb{R}^n)$, 则 $\|f\|_X = 0$ 当且仅当 $f = 0$ 几乎处处成立; 若 $f, g \in \mathscr{M}(\mathbb{R}^n)$ 且 $f = g$ 几乎处处成立, 则 $\|f\|_X \sim \|g\|_X$, 其中等价常数与 f 和 g 无关.

(ii) 如 [97, Remark 2.5(ii)] 中所述, 若本文在定义 4 中用任意有界可测集 E 或任意球 $B(x, r)$ 代替任意的伸缩球 $B \in \mathcal{B}$, 所得到的定义等价.

(iii) 由 [30, Theorem 2], 可知定义 4 的 (ii) 和 (iii) 可以得到任意的球拟 Banach 函数空间都是完备的.

接下来, 回顾球拟 Banach 函数空间 X 的 p-凸化和凹性的概念, 可见 [81, Definition 2.6].

ch-Debf **定义 6.** 设 X 是球拟 Banach 函数空间, $p\in(0,\infty)$.

(i) X 的p-凸化 X^p 定义为

$$X^p:=\{f\in\mathscr{M}(\mathbb{R}^n):\ |f|^p\in X\}$$

赋有拟范数 $\|f\|_{X^p}:=\||f|^p\|_X^{1/p}$.

(ii) 若存在正常数 C, 使得对于任意 $f_k\in\mathscr{M}(\mathbb{R}^n)$ 有

$$\sum_{k=1}^{\infty}\|f_k\|_X\le C\left\|\sum_{k=1}^{\infty}|f_k|\right\|_X$$

则称空间 X 具有凹性. 特别地, 当 $C=1$ 时, 称 X 具有严格凹性.

给定球 Banach 函数空间 X, 其伴随空间 (也称作 *Köthe* 对偶空间) X' 定义如下; 参见 [3, Chapter 1, Section 2] 或 [81, p. 9].

定义 7. 对于任意球 Banach 函数空间 X, 其相关空间 X'(也称为 *Köthe* 对偶空间) 定义为

$$X':=\left\{f\in\mathscr{M}(\mathbb{R}^n):\ \|f\|_{X'}:=\sup_{g\in X,\ \|g\|_X=1}\|fg\|_{L^1(\mathbb{R}^n)}<\infty\right\},$$

其中 $\|\cdot\|_{X'}$ 称为 $\|\cdot\|_X$ 的相关范数.

ch-bbf **注记 8.** 从 [81, Proposition 2.3] 可知, 若 X 是球 Banach 函数空间, 则其相关空间 X' 也是球 Banach 函数空间.

接下来, 回顾 X 的绝对连续拟范数的概念 (经典欧氏空间的情况, 参见 [95, Definition 3.2], 齐型空间的情况, 参见 [99, Definition 6.1]).

定义 9. 设 X 是球拟 Banach 函数空间. 若对于任意 $f\in X$, 当 $\{E_j\}_{j=1}^{\infty}$ 是一列可测集且满足对于任意 $j\in\mathbb{N}$ 有 $E_j\supset E_{j+1}$ 以及 $\bigcap_{j=1}^{\infty}E_j=\emptyset$, 且当 $j\to\infty$ 时, $\|f\mathbf{1}_{E_j}\|_X\downarrow 0$, 则称 f 在 X 中有绝对连续拟范数. 此外, 若对于任意 $f\in X$, f 在 X 中有绝对连续拟范数, 则称 X 有绝对连续拟范数.

接下来, 回顾 Hardy–Littlewood 极大算子的概念. 记 $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ 为 $\mathbb{R}^n$ 上的所有局部可积函数的集合. 对于 $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *Hardy–Littlewood* 极大算子 $\mathcal{M}(f)$ 的定义为, 对于任意 $x\in\mathbb{R}^n$,

$$\mathcal{M}(f)(x):=\sup_{k\in\mathbb{Z}}\sup_{y\in x+B_k}⨍_{y+B_k}|f(z)|\,dz=\sup_{x\in B\in\mathcal{B}}⨍_B|f(z)|\,dz,$$

其中 $\mathcal{B}$ 与 (3) 一致, 等式最后一项的上确界是对所有 $B \in \mathcal{B}$ 取得的. 对于任意给定的 $\alpha \in (0,\infty)$, *powered Hardy–Littlewood 极大算子* $\mathcal{M}^{(\alpha)}$ 的定义为, 对于任意 $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ 和 $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\alpha)}(f)(x) := \left\{\mathcal{M}\left(|f|^{\alpha}\right)(x)\right\}^{\frac{1}{\alpha}}.$$

本文还需要对给定的球拟 Banach 函数空间做出以下两个基本假设.

ch-Assum-1 **假设 10.** 设 X 是球拟 Banach 函数空间. 假设存在 $p_- \in (0,\infty)$, 使得对于任意 $p \in (0,p_-)$ 和 $u \in (1,\infty)$, 存在正常数 C, 依赖于 p 和 u, 使得对于任意 $\{f_k\}_{k=1}^{\infty} \subset \mathscr{M}(\mathbb{R}^n)$ 有

$$\left\|\left\{\sum_{k=1}^{\infty}\left[\mathcal{M}\left(f_k\right)\right]^u\right\}^{\frac{1}{u}}\right\|_{X^{\frac{1}{p}}} \le C\left\|\left\{\sum_{k=1}^{\infty}|f_k|^u\right\}^{\frac{1}{u}}\right\|_{X^{\frac{1}{p}}}.$$

注记 11. 设 X 是球 Banach 函数空间. 假设 $\mathcal{M}$ 在 X 上有界且 $\mathcal{M}$ 在 X' 上有界. 通过与 [26, Theorem 4.10] 证明中类似的论证, 能够得到 $\mathcal{M}$ 满足假设 10, 其中 $p_- = 1$.

在接下来的讨论中, 对于任意给定的 $p_- \in (0,\infty)$, 本文总是令

ch-underp (6)
$$\underline{p} := \min\{p_-, 1\}.$$

ch-Assum-2 **假设 12.** 设 $p_- \in (0,\infty)$, X 是球拟 Banach 函数空间. 假设存在 $\theta_0 \in (0,\underline{p})$, 其中 $\underline{p}$ 与(6)中的一致, 并且存在 $p_0 \in (\theta_0,\infty)$, 使得 X^{1/θ_0} 是球 Banach 函数空间, 并且对于任意 $f \in (X^{1/\theta_0})'$ 有

$$\left\|\mathcal{M}^{((p_0/\theta_0)')}(f)\right\|_{(X^{1/\theta_0})'} \le C\|f\|_{(X^{1/\theta_0})'},$$

其中 C 是正常数, 与 f 无关, 且 $\frac{1}{p_0/\theta_0} + \frac{1}{(p_0/\theta_0)'} = 1$.

接下来, 回顾 *Schwartz 函数* $\varphi \in C^{\infty}(\mathbb{R}^n)$ 是满足以下条件的函数, 对于任意 $k \in \mathbb{Z}_+$ 和任意多重指标 $\alpha \in \mathbb{Z}_+^n$,

ch-6.11.x1 (7)
$$\|\varphi\|_{\alpha,k} := \sup_{x\in\mathbb{R}^n}[\rho(x)]^k|\partial^{\alpha}\varphi(x)| < \infty.$$

记 $\mathcal{S}(\mathbb{R}^n)$ 为所有 *Schwartz 函数的集合*, 赋予其由 $\{\|\cdot\|_{\alpha,k}\}_{\alpha\in\mathbb{Z}_+^n, k\in\mathbb{Z}_+}$ 确定的拓扑. 则 $\mathcal{S}'(\mathbb{R}^n)$ 定义为 $\mathcal{S}(\mathbb{R}^n)$ 的对偶空间, 赋予其弱-$*$ 拓扑. 对于任意 $N \in \mathbb{Z}_+$,

$$\mathcal{S}_N(\mathbb{R}^n) := \left\{\varphi \in \mathcal{S}(\mathbb{R}^n) :\ \|\varphi\|_{\alpha,k} \le 1, |\alpha| \le N, k \le N\right\},$$

有如下等价关系,

$$\begin{aligned}&\varphi \in \mathcal{S}_N(\mathbb{R}^n)\\ &\quad\Longleftrightarrow\ \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}\max\{1,[\rho(x)]^N\}|\partial^{\alpha}\varphi(x)| \le 1.\end{aligned}$$

在接下来的讨论中, 对于任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ 和 $k \in \mathbb{Z}$, 令 $\varphi_k(\cdot) := b^{-k}\varphi(A^{-k}\cdot)$.

定义 13. 设 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ 和 $f \in \mathcal{S}'(\mathbb{R}^n)$. 对于任意 $x \in \mathbb{R}^n$, 关于 φ 的非切向极大函数$M_\varphi(f)$ 被定义为, 对于任意 $k \in \mathbb{Z}$, $y \in x + B_k$,

$$M_\varphi(f)(x) := \sup_{k\in\mathbb{Z},\, y\in x+B_k} |f * \varphi_k(y)|.$$

此外, 对于任意给定的 $N \in \mathbb{N}$, 关于 N 的非切向主极大函数$M_N(f)$ 定义为, 对于任意 $x \in \mathbb{R}^n$,

ch-M_N

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x). \tag{8}$$

在下文中, 本文引入了各向异性球 Campanato 函数空间的定义、$H_X^A(\mathbb{R}^n)$ 的原子特征和有限原子特征, 结合一些基本的数学分析结果. 建立了与球拟 Banach 函数空间 X 相关的各向异性 Hardy 空间 $H_X^A(\mathbb{R}^n)$ 的对偶定理. 为了陈述对偶定理, 首先按照 [97] 中的定义给出 $H_X^A(\mathbb{R}^n)$ 的定义. 在接下来的讨论中, 对于任意的 $\alpha \in \mathbb{R}$, 本文用 $\lfloor\alpha\rfloor$ 表示不大于 α 的最大整数.

ch-HXA **定义 14.** 设 A 是伸缩, X 是球拟 Banach 函数空间, 对于 $0 < p_- \in (0,\infty)$, 满足假设 10; 对于同一个的 p_-, $\theta_0 \in (0,\underline{p})$, 和 $p_0 \in (\theta_0,\infty)$, 满足假设 12, 其中 $\underline{p}$ 与(6)一致. 假设

ch-3.14.x1

$$N \in \mathbb{N} \cap \left[\left\lfloor\left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2, \infty\right). \tag{9}$$

则与 A 和 X 相关的各向异性 *Hardy* 空间 $H_{X,N}^A(\mathbb{R}^n)$ 定义为

$$H_{X,N}^A(\mathbb{R}^n) := \left\{f \in \mathcal{S}'(\mathbb{R}^n) :\ \|M_N(f)\|_X < \infty\right\},$$

其中 $M_N(f)$ 与(8)一致. 此外, 对于任意 $f \in H_{X,N}^A(\mathbb{R}^n)$, 定义

$$\|f\|_{H_{X,N}^A(\mathbb{R}^n)} := \|M_N(f)\|_X.$$

设 A 是伸缩, X 是定义 14 中的球拟 Banach 函数空间. 在后文中, 令

ch-NXA

$$N_{X,A} := \left\lfloor\left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2. \tag{10}$$

ch-s2r1 **注记 15.**

(i) 如 [97, Remark 2.17(i)] 中所述, 只要 $N \in \mathbb{N}\cap[N_{X,A},\infty)$, 空间 $H_{X,N}^A(\mathbb{R}^n)$ 就与 N 的选择无关. 在接下来的讨论中, 当 $N \in \mathbb{N}\cap[N_{X,A},\infty)$ 时, 本文将 $H_{X,N}^A(\mathbb{R}^n)$ 简写为 $H_X^A(\mathbb{R}^n)$.

(ii) 若 $A := 2\, I_{n\times n}$, 则 $H_X^A(\mathbb{R}^n)$ 与 Sawano 等人在 [81] 中引入的 $H_X(\mathbb{R}^n)$ 一致.

在接下来的讨论中, 对于任意 $d\in\mathbb{Z}_+$, $\mathcal{P}_d(\mathbb{R}^n)$ 表示 $\mathbb{R}^n$ 上所有次数不大于 d 的多项式的集合; 对于任意球 $B\in\mathcal{B}$ 和任意局部可积函数 g 在 $\mathbb{R}^n$ 上, 本文用 $P_B^d g$ 表示 g 的极小多项式, 其次数不大于 d, 表示 $P_B^d g$ 是 $\mathcal{P}_d(\mathbb{R}^n)$ 中的唯一多项式 f, 使得对于任意 $h\in\mathcal{P}_d(\mathbb{R}^n)$,

$$\int_B [g(x)-f(x)]h(x)\,dx=0.$$

下面, 本文引入与球拟 Banach 函数空间相关的各向异性球 Campanato 函数空间. 在接下来的讨论中, 用 $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ 表示 $\mathbb{R}^n$ 上的所有 q 阶局部可积函数的集合.

ch-LAXqds **定义 16.** 设 A 是伸缩, X 是球拟 Banach 函数空间, $q\in[1,\infty)$, $d\in\mathbb{Z}_+$, $s\in(0,\infty)$. 则与 A 和 X 相关的各向异性球 *Campanato* 函数空间 $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ 定义为所有使得下式有限的 $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ 的集合:

$$\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}:=\sup\left\|\left\{\sum_{i=1}^m\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\times\sum_{j=1}^m\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}\left|f(x)-P^d_{B^{(j)}}f(x)\right|^q\,dx\right]^{\frac{1}{q}}.$$

其中上确界在所有 $m\in\mathbb{N}$, $\{B^{(j)}\}_{j=1}^m\subset\mathcal{B}$ 和 $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$ 中取得.

ch-s2r2 **注记 17.** 设 A、X、q、d 和 s 与定义 16 一致.

(i) 若本文有基本假设 $\|\{\sum_{i=1}^m[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}]^s\mathbf{1}_{B^{(i)}}\}^{\frac{1}{s}}\|_X^{-1}\in(0,\infty)$, 则可以将定义 16 中的 m 取为 ∞; 参见下面的命题 18.

(ii) 显然, $\mathcal{P}_d(\mathbb{R}^n)\subset\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$. 实际上, $\|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}=0$ 当且仅当 $f\in\mathcal{P}_d(\mathbb{R}^n)$. 在本文中, 本文总是将某个 $f\in\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ 与 $\{f+P:\ P\in\mathcal{P}_d(\mathbb{R}^n)\}$ 等价.

(iii) 对于任意 $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, 定义

$$|||f|||_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}:=\sup\inf\left\|\left\{\sum_{i=1}^m\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac{1}{s}}\right\|_X^{-1}\times\sum_{j=1}^m\frac{\lambda_j|B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\fint_{B^{(j)}}|f(x)-P(x)|^q\,dx\right]^{\frac{1}{q}},$$

其中上确界的取法与定义 16 一致, 下确界取自所有 $P\in\mathcal{P}_d(\mathbb{R}^n)$. 然后, 类似于 [100, Lemma 2.5] 的证明, 易得 $|||\cdot|||_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}$ 是 $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ 的等价拟范数.

此外, 对于各向异性球 Campanato 函数空间 $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$, 本文有以下等价拟范数.

ch-s2p1 **命题 18.** 设 A、X、q、d 和 s 与定义 16 一致. 对于任意 $f \in L^q_{\rm loc}(\mathbb{R}^n)$, 定义

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} := \sup \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^s \mathbf{1}_{B^{(i)}} \right\}^{\frac{1}{s}} \right\|_X^{-1} \times \sum_{j\in\mathbb{N}} \frac{\lambda_j |B^{(j)}|}{\|\mathbf{1}_{B^{(j)}}\|_X} \left[⨍_{B^{(j)}} \left| f(x) - P^d_{B^{(j)}} f(x) \right|^q \, dx \right]^{\frac{1}{q}},$$

其中上确界在 $\{B^{(j)}\}_{j\in\mathbb{N}} \subset \mathcal{B}$ 和 $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0,\infty)$, 并满足以下条件

ch-s2e1 (11)
$$\left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X} \right]^s \mathbf{1}_{B^{(j)}} \right\}^{\frac{1}{s}} \right\|_X \in (0,\infty).$$

的全体中取得. 则, 对于任意 $f \in L^q_{\rm loc}(\mathbb{R}^n)$ 有

$$\widetilde{\|f\|}_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)} = \|f\|_{\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)}.$$

下面本文引入另一个与球拟 Banach 函数空间 X 相关联的各向异性的球 Campanato 函数空间 $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$.

ch-cqdA **定义 19.** 设 A 是伸缩, X 是球拟 Banach 函数空间, $q \in [1,\infty)$, $d \in \mathbb{Z}_+$. 则与 A 和 X 都相关的 *Campanato* 空间 $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ 定义为: 所有满足以下条件的 $f \in L^q_{\rm loc}(\mathbb{R}^n)$ 的集合:

$$\|f\|_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \frac{|B|}{\|\mathbf{1}_B\|_X} \left\{ ⨍_B \left| f(x) - P^d_B f(x) \right|^q \, dx \right\}^{\frac{1}{q}} < \infty,$$

其中上确界取在所有伸缩球 $B \in \mathcal{B}$ 中取得, $P^d_B f$ 表示 f 的次数不大于 d 的极小多项式.

ch-s2r3 **注记 20.** 设 A、X、q、d 和 s 与定义 16 一致.

(i) 从定义 16 和 19 可立即得到, $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n) \subset \mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$, 且此包含是连续的.

(ii) 对于任意 $f \in L^q_{\rm loc}(\mathbb{R}^n)$, 定义

$$|||f|||_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}} \inf_{P\in\mathcal{P}_d(\mathbb{R}^n)} \frac{|B|}{\|\mathbf{1}_B\|_X} \left[⨍_B |f(x) - P(x)|^q \, dx \right]^{\frac{1}{q}}.$$

则, 类似于 [100, Lemma 2.6], 可得 $|||\cdot|||_{\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)}$ 是 $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ 的一个等价拟范数.

下式是在本文中经常使用的基本不等式.

ch-basicine **引理 21.** 设 $\{a_i\}_{i\in\mathbb{N}} \subset [0,\infty)$. 若 $\alpha \in [1,\infty)$, 则

$$\left(\sum_{i\in\mathbb{N}} a_i \right)^\alpha \geq \sum_{i\in\mathbb{N}} a_i^\alpha.$$

以下命题表明, 若球拟 Banach 函数空间 X 是凹的, 并且 $s\in(0,1]$, 则 $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ 与在定义 19 中引入的 $\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$ 是相同的.

ch-s2p2 **命题 22.** 设 X 是一个凹的球拟 Banach 函数空间, $q\in[1,\infty)$, $d\in\mathbb{Z}_+$, $s\in(0,1]$. 此时有

ch-2.15.x3 (12)
$$\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)=\mathcal{L}^A_{X,q,d}(\mathbb{R}^n)$$

并且它们具有等价的拟范数.

接下来, 本文将建立 $\mathcal{L}^A_{X,q,d,s}(\mathbb{R}^n)$ 和 $H^A_X(\mathbb{R}^n)$ 之间的对偶关系. 为此, 首先回顾了来自 [97] 的定义, 即与球拟 Banach 函数空间 X 相关的各向异性 (X,q,d) 原子和有限原子 Hardy 空间 $H^{A,q,d}_{X,\mathrm{fin}}(\mathbb{R}^n)$.

ch-deffin **定义 23.** 设 A 是伸缩, X 是球拟 Banach 函数空间, 满足假设 10, 其中 $p_-\in(0,\infty)$; 以及假设 12, 其中 p_-, $\theta_0\in(0,\underline{p})$ $p_0\in(\theta_0,\infty)$, $\underline{p}$ 与(6)中一致. 假设 $N\in\mathbb{N}\cap[N_{X,A},\infty)$, 其中 $N_{X,A}$ 与(10)中一致. 进一步假设 $q\in(\max\{p_0,1\},\infty]$, 且

ch-def-d (13)
$$d\in\left[\left\lfloor\left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor,\infty\right)\cap\mathbb{Z}_+.$$

(i) *各向异性 (X,q,d) 原子* a 是一个在 $\mathbb{R}^n$ 上的可测函数, 满足以下条件:

- (i)$_1$ $\operatorname{supp} a:=\{x\in\mathbb{R}^n:\ a(x)\neq 0\}\subset B$, 其中 $B\in\mathcal{B}$, $\mathcal{B}$ 与(3)中一致;
- (i)$_2$ $\|a\|_{L^q(\mathbb{R}^n)}\le|B|^{\frac{1}{q}}\|\mathbf{1}_B\|_X^{-1}$;
- (i)$_3$ 对任意 $\gamma\in\mathbb{Z}^n_+$, 其中 $|\gamma|\le d$, 都有 $\int_{\mathbb{R}^n}a(x)x^\gamma\,dx=0$, 此处及后文中, 对任意 $\gamma:=\{\gamma_1,\dots,\gamma_n\}\in\mathbb{Z}^n_+$, $|\gamma|:=\gamma_1+\cdots+\gamma_n$, $x^\gamma:=x_1^{\gamma_1}\cdots x_n^{\gamma_n}$.

(ii) 与 A 和 X 相关的*各向异性有限原子 Hardy 空间*$H^{A,q,d}_{X,\mathrm{fin}}(\mathbb{R}^n)$, 定义为所有 $f\in\mathcal{S}'(\mathbb{R}^n)$, 满足存在 $K\in\mathbb{N}$, 一列 $\{\lambda_i\}_{i=1}^K\subset(0,\infty)$, 以及一列支集在 $\{B^{(i)}\}_{i=1}^K\subset\mathcal{B}$ 中的各向异性 (X,q,d) 原子列 $\{a_i\}_{i=1}^K$, 使得

$$f=\sum_{i=1}^K\lambda_i a_i$$

的集合. 此外, 对任意 $f\in H^{A,q,d}_{X,\mathrm{fin}}(\mathbb{R}^n)$, 定义

$$\|f\|_{H^{A,q,d}_{X,\mathrm{fin}}(\mathbb{R}^n)}:=\inf\left\|\left\{\sum_{i=1}^K\left[\frac{\lambda_i\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X,$$

其中下确界是在上述分解的所有情况中取得的.

设 A 是伸缩, X 与定义 23 一致. 后文中, 令

ch-dxa (14)
$$d_{X,A}:=\left\lfloor\left(\frac{1}{\theta_0}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor.$$

要建立 $H_X^A(\mathbb{R}^n)$ 的对偶定理, 本文需要其原子特征和有限原子特征如下, 它们分别是 [97, Theorem 4.2 and Lemma 7.2] 和 [97, Theorem 5.4] 的简单推论.

ch-s2l1 **引理 24.** 设 A、X、q、d 和 θ_0 与定义 23 中一致. 进一步假设 X 有绝对连续拟范数, $\{a_j\}_{j\in\mathbb{N}}$ 是一列支集分别在伸缩球 $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ 中的各向异性 (X,q,d)-原子列, $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$, 且满足

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right]^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X<\infty.$$

则级数 $f:=\sum_{j\in\mathbb{N}}\lambda_j a_j$ 在 $H_X^A(\mathbb{R}^n)$ 中收敛, $f\in H_X^A(\mathbb{R}^n)$, 且存在一个与 f 无关的正常数 C, 使得

$$\|f\|_{H_X^A(\mathbb{R}^n)}\le C\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right]^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X$$

成立.

ch-finatomth **引理 25.** 设 A、X、q、d、θ_0 和 p_0 与定义 23 中一致.

(i) 若 $q\in(\max\{p_0,1\},\infty)$, 则 $\|\cdot\|_{H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)}$ 和 $\|\cdot\|_{H_X^A(\mathbb{R}^n)}$ 在 $H_{X,\,\mathrm{fin}}^{A,q,d}(\mathbb{R}^n)$ 上是等价的拟范数;

(ii) $\|\cdot\|_{H_{X,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)}$ 和 $\|\cdot\|_{H_X^A(\mathbb{R}^n)}$ 在 $H_{X,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ 上是等价的拟范数, 其中 $\mathcal{C}(\mathbb{R}^n)$ 表示 $\mathbb{R}^n$ 上所有连续函数的集合.

建立对偶定理还需要以下结论.

ch-atomch2 **命题 26.** 设 A、X 和 d 与定义 23 一致. 则集合 $H_{X,\mathrm{fin}}^{A,\infty,d}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ 在 $H_X^A(\mathbb{R}^n)$ 中是稠密的.

下定理表明 $H_X^A(\mathbb{R}^n)$ 的对偶空间是 $\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$.

ch-s2t1 **定理 27.** 设 A、X、q、d 和 θ_0 与定义 23 一致. 进一步假设 X 具有绝对连续拟范数. 则 $H_X^A(\mathbb{R}^n)$ 的对偶空间, 记为 $(H_X^A(\mathbb{R}^n))^*$, 在以下意义下是 $\mathcal{L}_{X,q',d,\theta_0}^A(\mathbb{R}^n)$, 其中 $1/q+1/q'=1$:

(i) 设 $g \in \mathcal{L}^{A}_{X,q',d,\theta_0}(\mathbb{R}^n)$. 则对任意 $f \in H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$, 定义于 $H^{A,q,d}_{X,\,\mathrm{fin}}(\mathbb{R}^n)$ 上的线性泛函

ch-s2t1e1 (15)
$$L_g:\ f \to L_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\,dx$$

在 $H^A_X(\mathbb{R}^n)$ 上有有界延拓.

(ii) 相反地, 对于 $H^A_X(\mathbb{R}^n)$ 上的任意连续线性泛函, 都存在唯一的 $g \in \mathcal{L}^{A}_{X,q',d,\theta_0}(\mathbb{R}^n)$, 使其可表示为 (15).

此外, $\|g\|_{\mathcal{L}^{A}_{X,q',d,\theta_0}(\mathbb{R}^n)} \sim \|L_g\|_{(H^A_X(\mathbb{R}^n))^*}$, 其中等价常数与 g 无关.

作为定理 27 的推论, 有以下各向异性球 Campanato 函数空间的等价性.

ch-s2c1 **推论 28.** 设 A、X、d、θ_0 和 p_0 与定理 27 一致, 且当 $p_0 \in (0,1)$ 时 $q \in [1,\infty)$, 或当 $p_0 \in [1,\infty)$ 时 $q \in [1,p_0')$. 则

$$\mathcal{L}^{A}_{X,1,d_{X,A},\theta_0}(\mathbb{R}^n) = \mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)$$

并赋有等价拟范数, 其中 $d_{X,A}$ 与 (14) 一致.

ch-3.24.x1 **注记 29.** (i) 若 $A := 2\,I_{n\times n}$, 则定理 27 和推论 28 分别在 [103, Theorem 3.14 和 Corollary 3.15] 中得到.

(ii) 最近, Yan 等人在 [99, Theorem 6.6] 中得到了与给定的齐型空间 $\mathcal{X}$ 上的球拟 Banach 函数空间 $Y(\mathcal{X})$ 相关联的 Hardy 空间 $H_Y(\mathcal{X})$ 的对偶定理. 需要指出, 由于一般齐型空间 $\mathcal{X}$ 中不存在线性结构, 因此无法在 $\mathcal{X}$ 上引入 Schwartz 函数和多项式. 实际上, 在 [99] 中, 任何原子都只有零阶消失矩, 而定理 27 中的原子则具有消失矩, 其阶数为 $d \in [d_{X,A},\infty)\cap\mathbb{N}$, 其中 $d_{X,A}$ 与 (14) 一致. 因此, 尽管 $(\mathbb{R}^n,\rho,dx)$ 是一个齐型空间, 定理 27 不能由 [99, Theorem 6.6] 推导出来, 实际上, 它们互不包含对方.

二、$H^A_X(\mathbb{R}^n)$ 的各向异性 Littlewood–Paley 函数特征及 Fourier 变换

ch-radialM **定义 30.** 设 $\varphi \in \mathcal{S}(\mathbb{R}^n)$ 且 $f \in \mathcal{S}'(\mathbb{R}^n)$. 对于任意 $x \in \mathbb{R}^n$, 定义 f 关于 φ 的各向异性径向极大函数$M^0_\varphi(f)$ 为:

$$M^0_\varphi(f)(x) := \sup_{k\in\mathbb{Z}} |f*\varphi_k(x)|.$$

此外, 对于任意给定的 $N \in \mathbb{N}$, 定义 $f \in \mathcal{S}'(\mathbb{R}^n)$ 的各向异性径向主极大函数$M^0_N(f)$ 为:

$$M^0_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M^0_\varphi(f)(x).$$

接下来, 对于任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, 本文定义 $\widehat{\varphi}$ 为:

$$\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi \imath x\cdot\xi}\, dx,$$

其中 $\imath := \sqrt{-1}$ 且对于任意 $x := (x_1, \ldots, x_n), \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x \cdot \xi := \sum_{i=1}^n x_i \xi_i$. 对于任意 $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ 定义为:

$$\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle,$$

其中 $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

若对于任意 $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_k \to 0$ 在 $\mathcal{S}'(\mathbb{R}^n)$ 中随着 $k \to \infty$, 则称 $f \in \mathcal{S}'(\mathbb{R}^n)$ 在无穷远处弱消失 (参见, [36, p. 50]). 令 $\mathcal{C}_c^\infty(\mathbb{R}^n)$ 表示在 $\mathbb{R}^n$ 上有紧支集的无穷可微函数的全体. 以下的 Calderón 再生公式来自于 [7, Proposition 2.14].

ch-s4l1 **引理 31.** 设 $d \in \mathbb{Z}_+$, A 是伸缩. 假设 $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ 满足以下条件:

ch-s4l1e1 (16)
$$\operatorname{supp} \phi \subset B_0;\ \int_{\mathbb{R}^n} x^\gamma \phi(x)\, dx = 0 \text{ 对于任意 } \gamma \in \mathbb{Z}_+^n \text{ 且 } |\gamma| \le d \text{成立};$$

且存在正常数 C 使得

ch-s4l1e2 (17)
$$\text{当 } \xi \in \left\{x \in \mathbb{R}^n:\ (2||A||)^{-1} \le \rho(x) \le 1\right\} \text{时, 有} \left|\widehat{\phi}(\xi)\right| \ge C.$$

则存在一个 $\psi \in \mathcal{S}(\mathbb{R}^n)$ 满足以下条件:

(i) $\operatorname{supp} \widehat{\psi}$ 是紧的且远离原点;

(ii) 对于任意 $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ 有

$$\sum_{j\in\mathbf{Z}} \widehat{\psi}\left((A^*)^j \xi\right) \widehat{\phi}\left((A^*)^j \xi\right) = 1,$$

其中 A^* 表示 A 的伴随矩阵.

此外, 对于任意 $f \in \mathcal{S}'(\mathbb{R}^n)$, 若 f 在无穷远处弱消失, 则

$$f = \sum_{j\in\mathbb{Z}} f * \psi_j * \phi_j \text{ 在 } \mathcal{S}'(\mathbb{R}^n) \text{ 中成立}.$$

以下是各向异性 Lusin 面积函数、各向异性 Littlewood–Paley g 函数、各向异性 Littlewood–Paley $g_\lambda^*(f)$ 函数的定义。这些定义由 [76, Definition 2.6] 引进.

ch-de4.1 **定义 32.** 设 $\phi \in \mathcal{S}(\mathbb{R}^n)$ 与引理 31 一致. 对于任意 $f \in \mathcal{S}'(\mathbb{R}^n)$ 和取定的 $\lambda \in (0, \infty)$, 定义各向异性 *Lusin* 面积函数 $S(f)$, 各向异性 *Littlewood–Paley* g-函数 $g(f)$, 以及各向异性 *Littlewood–Paley* g_λ^*-函数如下:

ch-deaf (18)
$$S(f)(x) := \left[\sum_{k\in\mathbb{Z}} b^{-k} \int_{x+B_k} |f * \phi_k(y)|^2\, dy\right]^{\frac{1}{2}},$$

ch-deag
$$g(f)(x):=\left[\sum_{k\in\mathbb{Z}}\left|f*\phi_k(x)\right|^2\right]^{\frac{1}{2}}, \tag{19}$$

deaglambda
$$g_\lambda^*(f)(x):=\left[\sum_{k\in\mathbb{Z}}e^{-2\pi\lambda k}\left|f*\phi_k(x)\right|^2\right]^{\frac{1}{2}}. \tag{20}$$

下面定理用各向异性 Lusin 面积函数、各向异性 Littlewood–Paley g-函数和各向异性 Littlewood–Paley g_λ^*-函数来刻画 $H_X^A(\mathbb{R}^n)$.

ch-s4t1 **定理 33.** 设 A 是伸缩, X 是球拟 Banach 函数空间, 对于 $0<p_-\in(0,\infty)$, 满足假设 10; 对于同一个的 p_-, $\theta_0\in(0,\underline{p})$, 和 $p_0\in(\theta_0,\infty)$, 满足假设 12, 其中 $\underline{p}$ 与(6)一致; 当且仅当 $f\in\mathcal{S}'(\mathbb{R}^n)$, f 在无穷远处弱消失, 并且 $\|S(f)\|_X<\infty$ 时, $f\in H_X^A(\mathbb{R}^n)$. 此外, 对于任意 $f\in H_X^A(\mathbb{R}^n)$ 有

$$\|S(f)\|_X\sim\|f\|_{H_X^A(\mathbb{R}^n)},$$

其中等价常数与 f 无关.

ch-s4t1' **定理 34.** 设 A 和 X 与定理 33 一致. 则当且仅当 $f\in\mathcal{S}'(\mathbb{R}^n)$, f 在无穷远处弱消失, 并且 $\|g(f)\|_X<\infty$ 时, $f\in H_X^A(\mathbb{R}^n)$. 此外, 对于任意 $f\in H_X^A(\mathbb{R}^n)$ 有

$$\|g(f)\|_X\sim\|f\|_{H_X^A(\mathbb{R}^n)},$$

其中等价常数与 f 无关.

此外, 由定理 33 和定理 34, 以及类似于 [17, Theorem 4.11] 中使用的证明方法, 易得以下结果.

ch-s4t1'' **定理 35.** 设 A、X 和 θ_0 与定理 33 一致, $\lambda\in(\max\{1,2/r_+\},\infty)$, 其中

ch-3.25.x1
$$r_+:=\sup\left\{\theta_0\in(0,\infty):\ \text{存在 } p_0\in(\theta_0,\infty) \text{ 使得} X \text{ 满足假设 } 12\right\}. \tag{21}$$

则当且仅当 $f\in\mathcal{S}'(\mathbb{R}^n)$, f 在无穷远处弱消失, 并且 $\|g_\lambda^*(f)\|_X<\infty$ 时, $f\in H_X^A(\mathbb{R}^n)$. 此外, 对于任意 $f\in H_X^A(\mathbb{R}^n)$ 有

$$\|g_\lambda^*(f)\|_X\sim\|f\|_{H_X^A(\mathbb{R}^n)},$$

其中等价常数与 f 无关.

为了证明定理 33, 本文首先提出以下结论, 它表明了引理 31 中由不同的 ϕ 定义的各向异性 Lusin 面积函数的 $\|\cdot\|_X$ 拟范数等价.

ch-s4t2 **定理 36.** 设 A 和 X 与定理 33 一致, $\phi,\psi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ 满足 (16) 和 (17). 则对于任意在无穷远处弱消失的 $f\in\mathcal{S}'(\mathbb{R}^n)$ 有

$$\|S_\phi(f)\|_X\sim\|S_\psi(f)\|_X,$$

其中 $S_\phi(f)$ 与 (18) 一致, $S_\psi(f)$ 与 (18) 中的 ϕ 替换为 ψ, 并且等价常数与 f 无关.

为了证明定理 36, 本文需要以下引理, 它是来源于 [19, Theorem 11] 的 [7, Lemma2.3].

ch-defdya **引理 37.** 设 A 是伸缩, 则存在一族开子集

$$\mathcal{Q}:=\left\{Q_\alpha^k\subset\mathbb{R}^n:\ k\in\mathbb{Z},\alpha\in I_k\right\},$$

其中 I_k 是指标集, 满足以下条件:

(i) 对于每个固定的 k 有 $|\mathbb{R}^n\setminus\bigcup_\alpha Q_\alpha^k|=0$, 并且对于任意 $\alpha\neq\beta$ 有 $Q_\alpha^k\cap Q_\beta^k=\emptyset$;

(ii) 对于任意 α,β,k,ℓ, 若 $\ell\geq k$, 则 $Q_\alpha^k\cap Q_\beta^\ell=\emptyset$ 或 $Q_\alpha^\ell\subset Q_\beta^k$;

(iii) 对于每个 (ℓ,β) 和 $k<\ell$, 存在唯一的 α 使得 $Q_\beta^\ell\subset Q_\alpha^k$;

(iv) 存在某个负整数 v 和正整数 u, 对于任意 $k\in\mathbb{Z}$ 和 $\alpha\in I_k$, 存在 $x_{Q_\alpha^k}\in Q_\alpha^k$, 满足对于任意 $x\in Q_\alpha^k$ 有

$$x_{Q_\alpha^k}+B_{vk-u}\subset Q_\alpha^k\subset x+B_{vk+u}.$$

接下来, 为方便起见, 本文称引理 37 中的 $\mathcal{Q}:=\{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in I_k}$ 为二进方体, 并将 k 称为二进方体 Q_α^k 的层, 记为 $\ell(Q_\alpha^k)$.

以下的技术性引理也是必要的, 它是 [47, Lemma 6.9].

ch-s4l2 **引理 38.** 设 d 与 (13) 一致, v 和 u 与引理 37(iv) 一致,

$$\eta\in\left(\frac{\ln b}{\ln b+(d+1)\ln\lambda_-},1\right].$$

则存在正常数 C, 对于任意 $k,i\in\mathbb{Z}$, $\{c_Q\}_{Q\in\mathcal{Q}}\subset[0,\infty)$ (其中 $\mathcal{Q}$ 与引理 37 中一致), 以及 $x\in\mathbb{R}^n$ 有

$$\begin{aligned}&\sum_{\ell(Q)=\lceil\frac{k-u}{v}\rceil}|Q|\frac{b^{(k\vee i)(d+1)\frac{\ln\lambda_-}{\ln b}}}{[b^{(k\vee i)}+\rho(x-z_Q)]^{(d+1)\frac{\ln\lambda_-}{\ln b}+1}}c_Q\\&\quad\leq Cb^{-[k-(k\vee i)](\frac{1}{\eta}-1)}\left\{\mathcal{M}\left[\sum_{\ell(Q)=\lceil\frac{k-u}{v}\rceil}(c_Q)^\eta\mathbf{1}_Q\right](x)\right\}^{\frac{1}{\eta}},\end{aligned}$$

其中 $\ell(Q)$ 表示 Q 的层, $z_Q\in Q$, 对于任意 $k,i\in\mathbb{Z}$, $k\vee i:=\max\{k,i\}$.

接下来, 本文回顾一下与伸缩 A 相关的各向异性 Muckenhoupt 权函数的定义, 该定义在 [6, Definition 2.4] 中引入.

定义 39. 设 A 是伸缩, $p\in[1,\infty)$, w 是 $\mathbb{R}^n$ 上的非负可测函数. 则函数 w 被称为*各向异性 Muckenhoupt 权函数*, 若存在正常数 C, 当 $p\in(1,\infty)$ 时有

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{⨍_{x+B_k} w(y)\,dy\right\}\left\{⨍_{x+B_k}[w(y)]^{-\frac{1}{p-1}}\,dy\right\}^{p-1}\le C$$

或者, 当 $p=1$ 时有

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}}\left\{⨍_{x+B_k} w(y)\,dy\right\}\left\{\operatorname{ess}\sup_{y\in x+B_k}[w(y)]^{-1}\right\}\le C.$$

满足条件的各向异性 Muckenhoupt 权函数全体记为 $\mathcal{A}_p(A):=\mathcal{A}_p(\mathbb{R}^n,A)$. 此外, 上述条件下的最小常数 C 记为 $C_{p,A,n}(w)$.

容易证明, 若 $1\le p\le q\le\infty$, 则 $\mathcal{A}_p(A)\subset\mathcal{A}_q(A)$. 令 $\mathcal{A}_\infty(A):=\bigcup_{q\in[1,\infty)}\mathcal{A}_q(A)$. 对于任意给定的 $w\in\mathcal{A}_\infty(A)$, 定义 w 的*临界指标* q_w 为:

ch-qw (22)
$$q_w:=\inf\{p\in[1,\infty):\ w\in\mathcal{A}_p(A)\}.$$

显然, $q_w\in[1,\infty)$. 由逆 Hölder 不等式 (参见, 例如, [50, Theorem 1.2]) 得出, 对于任意 $p\in(1,\infty)$ 和 $w\in\mathcal{A}_p(A)$, 存在 $\epsilon\in(0,p-1]$ 有 $w\in\mathcal{A}_{p-\epsilon}(A)$. 因此, 若 $q_w\in(1,\infty)$, 则 $w\notin\mathcal{A}_{q_w}(A)$. 此外, Johnson 和 Neugebauer [54, p. 254] 给出了一个例子, $A:=2I_{n\times n}$, $w\notin\mathcal{A}_1(A)$, 但 $q_w=1$.

对于任意非负局部可积函数 w 和任意 Lebesgue 可测集 E, 令

$$w(E):=\int_E w(x)\,dx.$$

对于任意给定的 $p\in(0,\infty)$, 记 $L^p_w(\mathbb{R}^n)$ 为*所有可测函数* f 在 $\mathbb{R}^n$ 上的集合, 满足

$$\|f\|_{L^p_w(\mathbb{R}^n)}:=\left\{\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx\right\}^{\frac{1}{p}}<\infty.$$

此外, 令 $L^\infty_w(\mathbb{R}^n):=L^\infty(\mathbb{R}^n)$. 显然, $L^p_w(\mathbb{R}^n)$ 是球拟 Banach 函数空间, 而它却不是拟 Banach 函数空间 (参见, 例如, [81, p. 86]).

为了证明定理 33, 本文需要以下几个技术引理. 因为 $(\mathbb{R}^n,\rho,dx)$ 是一个特殊的齐型空间, 引理 40 是 [99, Lemma 4.9] (也可参见 [82, (4.6)]) 的直接推论, 引理 41 与 [4, p. 21, Theorem 4.5] 类似.

ch-embed **引理 40.** 设 A、X 和 θ_0 与定理 33 一致, $x_0\in\mathbb{R}^n$. 存在一个 $\epsilon\in(0,1)$, 使得 X 连续嵌入到 $L^{\theta_0}_w(\mathbb{R}^n)$ 中, 其中 $w:=[\mathcal{M}(\mathbf{1}_{x_0+B_0})]^\epsilon$, B_0 形如 (2) 取 $k=0$ 的情形.

ch-inclu **引理 41.** 设 A 和 X 与定理 33 一致, 有 $H_X^A(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, 且该包含关系是连续的.

结合引理 40 和引理 41, 可以得到了 $H_X^A(\mathbb{R}^n)$ 的以下性质.

ch-HXAvanish **引理 42.** 设 A 和 X 与定理 33 一致, $f \in H_X^A(\mathbb{R}^n)$, 有 f 弱无穷远处消失.

为了证明定理 33, 本文还需要以下引理, 其证明与 [69, Lemma 4.2] 类似.

ch-s4l3 **引理 43.** 设 A、X、θ_0 和 p_0 与定理 33 一致, $q \in (\max\{p_0, 1\}, \infty]$, $k_0 \in \mathbb{Z}$, $\varepsilon \in (0, \infty)$. 假设 $\{\lambda_i\}_{i\in\mathbb{N}} \subset [0, \infty)$, $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathcal{B}$, 以及 $\{m_i^{(\varepsilon)}\}_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ 满足, 对于任意 $\varepsilon \in (0, \infty)$ 和 $i \in \mathbb{N}$,

$$\mathrm{supp}\ m_i^{(\varepsilon)} := \left\{x \in \mathbb{R}^n : \ m_i^{(\varepsilon)} \neq 0\right\} \subset A^{k_0} B^{(i)},$$

$$\|m_i^{(\varepsilon)}\|_{L^q(\mathbb{R}^n)} \leq \frac{|B^{(i)}|^{\frac{1}{q}}}{\|\mathbf{1}_{B^{(i)}}\|_X},$$

和

$$\left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X < \infty.$$

则有

$$\left\| \liminf_{\varepsilon\to 0^+} \left[\sum_{i\in\mathbb{N}} \left| \lambda_i m_i^{(\varepsilon)} \right|^{\theta_0} \right]^{\frac{1}{\theta_0}} \right\|_X \leq C \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X} \right]^{\theta_0} \right\}^{\frac{1}{\theta_0}} \right\|_X,$$

其中 C 是与 λ_i、$B^{(i)}$、$m_i^{(\varepsilon)}$ 和 ε 无关的正常数.

ch-3.24.x3 **注记 44.** (i) 若 $A := 2\, I_{n\times n}$, 则定理 33、34 和 35 是由 [17, Theorems 4.9, 4.11, and 4.13] (也可参见 [81, Theorem 3.21] 和 [95, Theorem 2.10]) 得到的.

(ii) 正如在注记 29(ii) 中提到的那样, 尽管 $(\mathbb{R}^n, \rho, dx)$ 是齐型空间, 定理 33、34 和 35 不能从 [98, Theorems 4.11, 5.1, and 5.3] 中推出, 实际上, 它们不能相互包含.

下面, 本文研究 $f \in H_X^A(\mathbb{R}^n)$ 的 Fourier 变换. 回顾一下, 对于任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, 它的 *Fourier 变换*, 记为 $\mathscr{F}(\varphi)$ 或 $\widehat{\varphi}$, 定义为, 对于任意 $\xi \in \mathbb{R}^n$,

$$\mathscr{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi\imath x\cdot\xi}\, dx,$$

此处及后文中, $\imath := \sqrt{-1}$, 对于任意 $x := (x_1, \ldots, x_n), \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x \cdot \xi := \sum_{i=1}^n x_i \xi_i$. 对于任意 $f \in \mathcal{S}'(\mathbb{R}^n)$, $\widehat{f}$ 定义为, 对于任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle$; 同时, 对于任意 $f \in \mathcal{S}(\mathbb{R}^n)$[或 $\mathcal{S}'(\mathbb{R}^n)$], $f^\vee$ 表示其 *Fourier 逆变换*, 定义为, 对于任意 $\xi \in \mathbb{R}^n$, $f^\vee(\xi) := \widehat{f}(-\xi)$[或者对于任意 $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\langle f^\vee, \varphi \rangle := \langle f, \varphi^\vee \rangle$].

关于 $f \in H_X^A(\mathbb{R}^n)$ 的 Fourier 变换, 有如下结果.

ch-c3s2t1 **定理 45.** 设 A 是伸缩, 设 A 是伸缩, X 是球拟 Banach 函数空间, 对于 $0 < p_- \in (0,\infty)$, 满足假设 10; 对于同一个的 p_-, $\theta_0 \in (0, \underline{p})$, 和 $p_0 \in (\theta_0, \infty)$, 满足假设 12, 其中 $\underline{p}$ 与(6)一致; 进一步假设存在 $q_0 \in [\theta_0, 1]$ 使得:

(i) 对于任意非负可测函数列 $\{f_k\}_{k=1}^\infty$ 有

ch-6.9.x1 (23)
$$\sum_{k=1}^\infty \|f_k\|_{X^{\frac{1}{q_0}}} \lesssim \left\|\sum_{k=1}^\infty f_k\right\|_{X^{\frac{1}{q_0}}},$$

此处隐含的正常数与 $\{f_k\}_{k=1}^\infty$ 无关;

(ii) 对任意 $B \in \mathcal{B}$, 其中 $\mathcal{B}$ 如 (3) 定义, 有

ch-one_B_X (24)
$$\|\mathbf{1}_B\|_X \gtrsim \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\},$$

此处隐含的正常数与 B 无关.

则对任意 $f \in H_X^A(\mathbb{R}^n)$, 存在 $\mathbb{R}^n$ 上的连续函数 F 使得

ch-6.10.x3 (25)
$$\widehat{f} = F \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

并且存在正常数 C, 仅依赖于 A 和 X, 使得对任意 $x \in \mathbb{R}^n$ 有

ch-c3s2e1 (26)
$$|F(x)| \le C\|f\|_{H_X^A(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1}, [\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\},$$

此处 ρ_* 与定义 3 一致, 将其中的 A 替换成其转置矩阵 A^T.

ch-c3s2re1 **注记 46.** (i) 若 $A := 2\, I_{n\times n}$, 则定理 45 可以在 [46, Theorem 2.1] 中找到.

(ii) 对任意给定的可测集 $E \subset \mathbb{R}^n$ 和任何给定的 $p \in (0,\infty)$, *Lebesgue* 空间 $L^p(E)$ 的定义为

ch-6.11.y1 (27)
$$L^p(E) := \left\{f \text{ 是 } E \text{ 上的可测函数}: \|f\|_{L^p(E)} := \left[\int_E |f(x)|^p\,dx\right]^{1/p} < \infty\right\}.$$

假设 A 是伸缩, $p \in (0,1)$, 以及

$$N \in \mathbb{N} \cap \left[\left\lfloor\left(\frac{1}{p} - 1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor + 2, \infty\right).$$

由 [99, Remarks 2.7(i) and 4.21(i)], 能够得出 $L^p(\mathbb{R}^n)$ 满足定义 14 的所有假设, 其中 $X := L^p(\mathbb{R}^n)$, $p_- \in (0,p]$, $\theta_0 \in (0, p_-)$, 以及 $p_0 \in (p, \infty)$. 此外, 取 $q_0 \in (p, 1]$. 根据 (27), 对任意非负可测函数 $\{f_k\}_{k=1}^\infty$ 和任意 $B \in \mathcal{B}$ 有

$$\sum_{k=1}^\infty \|f_k\|_{L^{\frac{p}{q_0}}(\mathbb{R}^n)} \le \left\|\sum_{k=1}^\infty f_k\right\|_{L^{\frac{p}{q_0}}(\mathbb{R}^n)}$$

以及

$$\|\mathbf{1}_B\|_{L^p(\mathbb{R}^n)} = |B|^{\frac{1}{p}} > \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\}.$$

因此, $L^p(\mathbb{R}^n)$ 满足定理 45 的所有假设, 其中 $X := L^p(\mathbb{R}^n)$. 在这种情况下, 定理 45 可以在 [9, Theorem 1] 中找到.

(iii) 如 [46, Remark 2.1(ii)] 中提到的那样, (26) 意味着函数 $f \in H_X^A(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ 具有一个消失矩. 这在某种程度上说明了某些情况下原子的消失矩的必要性.

为了证明定理 45, 本文需要做更多的准备工作. 设 A 是伸缩. 定义算子 D_A 为, 对任意 $f \in \mathscr{M}(\mathbb{R}^n)$,

$$D_A(f)(\cdot) := f(A\cdot).$$

通过一个基本的计算 (参见 [9, (3.1)]), 能够发现对任意 $k \in \mathbb{Z}$, $f \in L^1(\mathbb{R}^n)$, 以及 $x \in \mathbb{R}^n$ 有

ch-6.10.x1 (28)
$$\widehat{f}(x) = b^k \left(D_{A^*}^k \left(\mathscr{F}\left(D_A^k f\right)\right)\right)(x).$$

接下来, 回顾各向异性原子 Hardy 空间的定义, 这是在 [97, Definition 4.2] 中首次引入的.

ch-atom **定义 47.** 设 A, X, θ_0 和 p_0 与定义 14 一致. 进一步假设 $q \in (\max\{p_0, 1\}, \infty]$ 且

(29)
$$d \in \left[\left\lfloor\left(\frac{1}{\theta_0} - 1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor, \infty\right) \cap \mathbb{Z}_+.$$

各向异性原子 Hardy 空间 $H_{X,\mathrm{atom}}^{A,q,d}(\mathbb{R}^n)$ 定义为所有满足以下条件的 $f \in \mathcal{S}'(\mathbb{R}^n)$ 的全体: 存在一列 $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ 和一列支集在 $\{B^j\}_{j\in\mathbb{N}} \subset \mathcal{B}$ 上的各向异性 (X, q, d)-原子 $\{a_j\}_{j\in\mathbb{N}}$, 满足在 $\mathcal{S}'(\mathbb{R}^n)$ 中有

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j,$$

并且有

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{|\lambda_j|\mathbf{1}_{B^j}}{\|\mathbf{1}_{B^j}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X < \infty.$$

此外, 对任意 $f \in H_{X,\mathrm{atom}}^{A,q,d}(\mathbb{R}^n)$, 设

$$\|f\|_{H_{X,\mathrm{atom}}^{A,q,d}(\mathbb{R}^n)} := \inf\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{|\lambda_j|\mathbf{1}_{B^j}}{\|\mathbf{1}_{B^j}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X,$$

其中的下确界是在所有上述分解中取得的.

下面是 $H_X^A(\mathbb{R}^n)$ 的原子特征, 这是由 [97, Theorem 4.3] 得到的.

ch-c3s210 **引理 48.** 设 A, X, q 和 d 与定义 47 中一致. 则 $H_X^A(\mathbb{R}^n)=H_{X,\mathrm{atom}}^{A,q,d}(\mathbb{R}^n)$, 且具有等价拟范数.

通过类似于 [9, Lemma 4] 中使用的论证, 立即得到以下结论.

ch-c3s211 **引理 49.** 设 A, X, q 和 d 与定义 47 中一致, a 是一个支在 $x_0+B_{i_0}$ 上的各向异性 (X,q,d)-原子, 其中 $x_0\in\mathbb{R}^n$, $i_0\in\mathbb{Z}$. 则存在正常数 C, 使得对任意满足 $|\alpha|\le d$ 的 $\alpha\in\mathbb{Z}_+^n$ 和任意 $x\in\mathbb{R}^n$ 有

ch-5.30.x2
$$\left|\partial^\alpha\left(\mathscr{F}\left(D_A^{i_0}a\right)\right)(x)\right|\le C\left\|\mathbf{1}_{B_{i_0}}\right\|_X^{-1}\min\left\{1,|x|^{d-|\alpha|+1}\right\},\tag{30}$$

其中常数 C 与 a 无关.

利用引理 49, 可以得到了对各向异性 (X,q,d)-原子的一致估计, 这在定理 45 的证明中起着关键作用.

ch-c3s212 **引理 50.** 设 A、X、q、d 和 θ_0 与定义 47一致. 进一步假设 X 满足 (24), 其中 $q_0\in[\theta_0,1]$. 则存在正常数 C, 对任意各向异性 (X,q,d)-原子 a 和任意 $x\in\mathbb{R}^n$ 有

ch-c3s2e2
$$|\widehat{a}(x)|\le C\max\left\{[\rho_*(x)]^{\frac{1}{q_0}-1},\,[\rho_*(x)]^{\frac{1}{\theta_0}-1}\right\},\tag{31}$$

其中 ρ_* 与定理 45 中一致.

引理 50 的证明需要以下不等式, 是 [4, p. 11, Lemma 3.2].

ch-c3s213 **引理 51.** 设 A 是伸缩. 则存在正常数 C, 对任意 $x\in\mathbb{R}^n$ 有

$$\frac{1}{C}[\rho(x)]^{\ln(\lambda_-)/\ln b}\le|x|\le C[\rho(x)]^{\ln(\lambda_+)/\ln b}\quad \rho(x)\in(1,\infty)$$

以及

$$\frac{1}{C}[\rho(x)]^{\ln(\lambda_+)/\ln b}\le|x|\le C[\rho(x)]^{\ln(\lambda_-)/\ln b}\quad \rho(x)\in[0,1].$$

引理 52 也在定理 45 的证明中使用.

ch-c3s214 **引理 52.** 设 A、X 和 θ_0 与定义 47一致. 进一步假设 X 满足 (23), 其中 $q_0\in[\theta_0,1]$. 则存在正常数 C, 对任意 $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ 和 $\{B^{(i)}\}_{i\in\mathbb{N}}\subset\mathcal{B}$ 有

$$\sum_{i\in\mathbb{N}}|\lambda_i|\le C\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^{\theta_0}\right\}^{\frac{1}{\theta_0}}\right\|_X.$$

作为定理 45 的应用, 本文首先证明了定理 53 中给出的函数 F 在原点具有更高阶的收敛性. 然后, 本文将 Hardy–Littlewood 不等式扩展到与球拟 Banach 函数空间相关的各向异性 Hardy 空间的情况 (参见定理 54).

ch-c3s3t1 **定理 53.** 设 A、X、q_0 和 ρ_* 如定理 45 一致. 则对于任何 $f\in H_X^A(\mathbb{R}^n)$, 都存在 $\mathbb{R}^n$ 上的连续函数 F, 使得在 $\mathcal{S}'(\mathbb{R}^n)$ 中有 $\widehat{f}=F$, 并且

ch-c3s3e1 (32)
$$\lim_{|x|\to 0^+}\frac{F(x)}{[\rho_*(x)]^{\frac{1}{\theta_0}-1}}=0.$$

作为定理 45 的另一个应用, 本文将 Hardy–Littlewood 不等式扩展到与球拟 Banach 函数空间相关的各向异性 Hardy 空间的情况如下.

ch-c3s3t2 **定理 54.** 设 A、X、θ_0 和 q_0 如定理 45 一致. 则对任意 $f\in H_X^A(\mathbb{R}^n)$, 都存在 $\mathbb{R}^n$ 上的连续函数 F, 使得在 $\mathcal{S}'(\mathbb{R}^n)$ 中有 $\widehat{f}=F$, 并且

ch-c3s3e6 (33)
$$\left[\int_{\mathbb{R}^n}|F(x)|^{q_0}\min\left\{[\rho_*(x)]^{q_0-\frac{q_0}{\theta_0}-1},\,[\rho_*(x)]^{q_0-2}\right\}dx\right]^{\frac{1}{q_0}}\le C\|f\|_{H_X^A(\mathbb{R}^n)},$$

其中 C 是只与 A 和 X 有关的正常数.

ch-6.11rem **注记 55.** (i) 若 $A:=2\,I_{n\times n}$, 则定理 53 和定理 54 分别在 [46, Theorems 2.2 and 2.3] 中得到.

(ii) 设 A 是伸缩, $p\in(0,1)$. 通过 Remark 46(ii), $L^p(\mathbb{R}^n)$ 满足定理 53 和定理 54 中的所有假设, 其中 $X:=L^p(\mathbb{R}^n)$. 在这种情况下, 定理 53 和定理 54 分别在 [9, Corollaries 6 and 8] 中得到.

三、$\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$ 的实变特征

在本节中, 利用在第一节中得到的对偶定理, 本文建立了各向异性球 Campanato 型函数空间 $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$ 的范数等价刻画. 这在建立 $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ 的 Carleson 测度刻画中起着重要作用.

ch-s3t1 **定理 56.** 设 A、X、q、d 和 θ_0 与推论 28 一致, 并且对于某个 $s\in(0,\theta_0)$ 有

ch-2.19.y2 (34)
$$\varepsilon\in\left(\frac{\ln b}{\ln(\lambda_-)}\left[\frac{2}{s}+d\frac{\ln(\lambda_+)}{\ln b}\right],\infty\right).$$

则以下陈述彼此等价:

(i) $f\in\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$;

(ii) $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ 并且

ch-2.19.y1 (35)

$$\|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}:=\sup\left\|\left\{\sum_{i=1}^m\left(\frac{\lambda_i}{\|\mathbf{1}_{x_i+B_{l_i}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_i+B_{l_i}}\right\}^{\frac{1}{\theta_0}}\right\|_X^{-1}\times\sum_{j=1}^m\frac{\lambda_j|x_j+B_{l_j}|}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\times\int_{\mathbb{R}^n}\frac{b^{\varepsilon l_j\frac{\ln(\lambda_-)}{\ln b}}|f(x)-P^d_{x_j+B_{l_j}}f(x)|}{b^{l_j[1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}]}+[\rho(x-x_j)]^{1+\varepsilon\frac{\ln(\lambda_-)}{\ln b}}}\,dx<\infty,$$

其中上确界是对所有 $m\in\mathbb{N}$, $\{x_j+B_{l_j}\}_{j=1}^m\subset\mathcal{B}$, $\{x_j\}_{j=1}^m\subset\mathbb{R}^n$, $\{l_j\}_{j=1}^m\subset\mathbb{Z}$, 以及 $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$ 取得的.

此外, 对任意 $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}^{A}_{X,q,d,\theta_0}(\mathbb{R}^n)}\sim\|f\|_{\mathcal{L}^{A,\varepsilon}_{X,1,d,\theta_0}(\mathbb{R}^n)}$$

其中等价常数与 f 无关.

为了证明定理 56, 本文需要以下技术性引理, 这是由点态估计 $\mathbf{1}_{x_j+B_{k_j+\ell}}\le b^\ell\mathcal{M}(\mathbf{1}_{x_j+B_{k_j}})$ 直接推导出的.

ch-s3l2 **引理 57.** 设 X 是一个球拟 Banach 函数空间, 满足假设 10, 其中 $p_-\in(0,\infty)$, 进一步假设 $\ell\in\mathbb{Z}_+$, $s\in(0,\min\{p_-,1\})$, 则存在正常数 C, 与 ℓ 和 s 都无关, 使得对任意序列 $\{x_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^n$ 和任意序列 $\{k_j\}_{j\in\mathbb{N}}\subset\mathbb{Z}$ 有

$$\left\|\sum_{j\in\mathbb{N}}\mathbf{1}_{x_j+B_{k_j+\ell}}\right\|_X\le Cb^{\frac{\ell}{s}}\left\|\sum_{j\in\mathbb{N}}\mathbf{1}_{x_j+B_{k_j}}\right\|_X,$$

其中, 对任意 $j\in\mathbb{N}$, B_{k_j} 与 (2) 一致.

本文还可以得到 $\mathcal{L}^A_{X,q,d,\theta_0}(\mathbb{R}^n)$ 的另一个等价刻画, 如下所示, 其证明是对定理 56 的轻微修改.

ch-s3t2 **定理 58.** 若 A、X、q、d、θ_0 和 ε 与定理 56 一致, 则定理 56 的结论在 m 替换为 ∞ 的情况下仍然成立, 其中上确界是对 $\{x_j+B_{l_j}\}_{j\in\mathbb{N}}\subset\mathcal{B}$, $\{x_j\}_{j\in\mathbb{N}}$, $\{l_j\}_{j\in\mathbb{N}}$, 和 $\{\lambda_j\}_{j\in\mathbb{N}}$ 满足

$$\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{x_j+B_{l_j}}\|_X}\right)^{\theta_0}\mathbf{1}_{x_j+B_{l_j}}\right\}^{\frac{1}{\theta_0}}\right\|_X\in(0,\infty)$$

的全体取得的.

ch-3.24.x2 **注记 59.** 若 $A:=2\,I_{n\times n}$, 则定理 56 和定理 58 分别在 [103, Theorems 4.1 and 4.4] 中得到.

应用前几节得到的结果, 本文建立了各向异性球 Campanato 函数空间 $\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$ 的 Carleson 测度刻画. 为此, 本文首先引入以下*各向异性 X-Carleson 测度*.

定义 60. 设 A 是伸缩, X 是球拟 Banach 函数空间, $s\in(0,\infty)$. $\mathbb{R}^n\times\mathbb{Z}$ 上的一个 Borel 测度 $d\mu$ 称为*各向异性 X–Carleson 测度*, 若其满足以下条件:

$$\begin{aligned}\|d\mu\|_X^{A,s}:=\sup&\left\|\left\{\sum_{i=1}^m\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac1s}\right\|_X^{-1}\\&\times\sum_{j=1}^m\left\{\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\right\}\\&<\infty,\end{aligned}$$

其中的上确界是对所有的 $m\in\mathbb{N}$, $\{B^{(j)}\}_{j=1}^m\subset\mathcal{B}$, 以及 $\{\lambda_j\}_{j=1}^m\subset(0,\infty)$ 取得的, 对于任意的 $j\in\{1,\dots,m\}$, $\widehat{B^{(j)}}$ 表示 $B^{(j)}$ 上的*帐篷*, 即

ch-hatB (36)
$$\widehat{B^{(j)}}:=\left\{(y,k)\in\mathbb{R}^n\times\mathbb{Z}:\ y+B_k\subset B^{(j)}\right\}.$$

对于各向异性 X–Carleson 测度, 本文有以下等价刻画.

ch-s5p1 **命题 61.** 设 A 是伸缩, X 是球拟 Banach 函数空间, $d\mu$ 是 $\mathbb{R}^n\times\mathbb{Z}$ 上的 Borel 测度, $s\in(0,\infty)$, 令

$$\begin{aligned}\widetilde{\|d\mu\|_X^{A,s}}:=\sup&\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac1s}\right\|_X^{-1}\\&\times\sum_{j\in\mathbb{N}}\left\{\frac{\lambda_j|B^{(j)}|^{\frac12}}{\|\mathbf{1}_{B^{(j)}}\|_X}\left[\int_{\widehat{B^{(j)}}}|d\mu(x,k)|\right]^{\frac12}\right\},\end{aligned}$$

其中的上确界是对 $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$ 和 $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\infty)$ 满足

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{\lambda_i}{\|\mathbf{1}_{B^{(i)}}\|_X}\right]^s\mathbf{1}_{B^{(i)}}\right\}^{\frac1s}\right\|_X\in(0,\infty).$$

的全体取得的. 则 $\widetilde{\|d\mu\|_X^{A,s}}=\|d\mu\|_X^{A,s}$.

接下来, 对于任意给定的 $k\in\mathbb{Z}$, 定义

$$\delta_k(j):=\begin{cases}1 & \text{当 } j=k,\\ 0 & \text{当 } j\neq k.\end{cases}$$

关于 Carleson 测度刻画的主要定理如下.

ch-s5t1 **定理 62.** 设 A, X, d 和 θ_0 与定义 23 中一致, $p_0\in(\theta_0,2)$, $\phi\in\mathcal{S}(\mathbb{R}^n)$ 是满足(16)和(17)的径向实值函数.

(i) 若 $h\in\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$, 则对于任意 $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$, $d\mu(x,k):=\sum_{\ell\in\mathbb{Z}}|\phi_\ell*h(x)|^2dx\,\delta_\ell(k)$ 是 $\mathbb{R}^n\times\mathbb{Z}$ 上的一个 X–Carleson 测度；此外, 存在一个与 h 无关的正常数 C, 使得

$$\|d\mu\|_X^{A,\theta_0}\le C\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}.$$

(ii) 若 $h\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, 且对于任意 $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$, $d\mu(x,k):=\sum_{\ell\in\mathbb{Z}}|\phi_\ell*h(x)|^2dx\,\delta_\ell(k)$ 是 $\mathbb{R}^n\times\mathbb{Z}$ 上的一个 X–Carleson 测度, 则 $h\in\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)$, 并且, 此外, 存在一个与 h 无关的正常数 C, 使得

$$\|h\|_{\mathcal{L}^A_{X,1,d,\theta_0}(\mathbb{R}^n)}\le C\,\|d\mu\|_X^{A,\theta_0}.$$

注记 63. (i) 注意, 若 X 是一个凹的球拟 Banach 函数空间, 命题 22, 定理 62 给出了 $\mathcal{L}^A_{X,1,d}(\mathbb{R}^n)$ 的 Carleson 测度刻画.

(ii) 若 $A:=2\,I_{n\times n}$, 则定理 62 已经在 [103, Theorem 5.3] 中得到.

为了证明定理 62, 需要与球拟 Banach 函数空间相关的各向异性帐篷空间及其原子分解. 本文首先回顾以下概念.

ch-defcone **定义 64.** 设 A 是伸缩, 对于任意 $x\in\mathbb{R}^n$, 令

$$\Gamma(x):=\{(y,k)\in\mathbb{R}^n\times\mathbb{Z}:\ y\in x+B_k\},$$

称其顶点为 $x\in\mathbb{R}^n$, 孔径为 1 的锥.

设 $\alpha\in(0,\infty)$. 对于任意可测函数 $F:\ \mathbb{R}^n\times\mathbb{Z}\to\mathbb{C}$ 和 $x\in\mathbb{R}^n$, 定义

$$\mathscr{A}(F)(x):=\left[\sum_{\ell\in\mathbb{Z}}b^{-\ell}\int_{\{y\in\mathbb{R}^n:\ (y,\ell)\in\Gamma(x)\}}|F(y,\ell)|^2\,dy\right]^{\frac12},$$

其中 $\Gamma(x)$ 与定义 64 一致. 若

$$\|F\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})}:=\|\mathscr{A}(F)\|_{L^p(\mathbb{R}^n)}<\infty,$$

则称可测函数 F 属于**各向异性帐篷空间** $T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})$, 其中 $p\in(0,\infty)$. 对于给定的球拟 Banach 函数空间 X, **各向异性 X-帐篷空间** $T_X^A(\mathbb{R}^n\times\mathbb{Z})$ 定义为所有在 $\mathbb{R}^n\times\mathbb{Z}$ 上, 满足 $\mathscr{A}(F)\in X$ 的可测函数 F 的全体, 并且自然地赋有拟范数:

$$\|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})}:=\|\mathscr{A}(F)\|_X.$$

接下来给出各向异性 (T_X,p)-原子的定义.

ch-s5d1 **定义 65.** 设 $p\in(1,\infty)$, A 是伸缩, X 是一个球拟 Banach 函数空间. 若存在球 $B\subset\mathcal{B}$, 使得

(i) $\operatorname{supp} a:=\{(x,k)\in\mathbb{R}^n\times\mathbb{Z}:\ a(x,k)\neq 0\}\subset\widehat{B}$, 其中 $\widehat{B}$ 由 (36) 中的 $B^{(j)}$ 替换为 B.

(ii) $\|a\|_{T_2^{A,p}(\mathbb{R}^n\times\mathbb{Z})}\le|B|^{1/p}/\|\mathbf{1}_B\|_X$.

则可测函数 $a:\ \mathbb{R}^n\times\mathbb{Z}\to\mathbb{C}$ 被称为一个**各向异性 (T_X,p)-原子**. 此外, 若对任意 $p\in(1,\infty)$, a 都是各向异性 (T_X,p)-原子, 则称 a 是一个**各向异性 (T_X,∞)-原子**.

可以得到以下关于各向异性 X-帐篷空间 $T_X^A(\mathbb{R}^n\times\mathbb{Z})$ 的原子分解.

ch-s5l1 **引理 66.** 设 A, X 和 θ_0 与定义 23 一致, $F:\ \mathbb{R}^n\times\mathbb{Z}\to\mathbb{C}$ 是一个可测函数. 若 $F\in T_X^A(\mathbb{R}^n\times\mathbb{Z})$, 则存在一列 $\{\lambda_j\}_{j\in\mathbb{N}}\subset[0,\infty)$, 一列 $\{B^{(j)}\}_{j\in\mathbb{N}}\subset\mathcal{B}$, 以及一列分别支集在 $\{\widehat{B^{(j)}}\}_{j\in\mathbb{N}}$ 上的各向异性 (T_X,∞)-原子 $\{A_j\}_{j\in\mathbb{N}}$, 使得对于几乎处处的 $(x,k)\in\mathbb{R}^n\times\mathbb{Z}$ 有

$$F(x,k)=\sum_{j\in\mathbb{N}}\lambda_jA_j(x,k),\ |F(x,k)|=\sum_{j\in\mathbb{N}}\lambda_j|A_j(x,k)|$$

并且

ch-s5e2 (37)
$$\left\|\left\{\sum_{j\in\mathbb{N}}\left(\frac{\lambda_j}{\|\mathbf{1}_{B^{(j)}}\|_X}\right)^{\theta_0}\mathbf{1}_{B^{(j)}}\right\}^{\frac{1}{\theta_0}}\right\|_X\lesssim\|F\|_{T_X^A(\mathbb{R}^n\times\mathbb{Z})},$$

其中隐含的正常数与 F 无关.

四、具体应用

在这一节中, 本文将定理 27、33、34、35、45、53、54、56、58 和 62 以及推论 28 应用于七个具体的球拟 Banach 函数空间, 分别是 Morrey 空间、Orlicz-slice 空间、Lorentz 空间、变指标 Lebesgue 空间、混合范数 Lebesgue 空间、加权 Lebesgue 空间和 Orlicz 空间. 特别地, 本文举例指出定理 45、53 和 54 不能应用于 Morrey 空间, 因为其范数缺乏凹性.

h-s6-appl1

Morrey 空间

经典的 Morrey 空间 $M_q^p(\mathbb{R}^n)$, 其中 $0<q\le p<\infty$, 最初由 Morrey 在 1938 年引入, 是调和分析和偏微分方程的理论基础. 此后, 在不同底空间上各种形式的 Morrey 空间被广泛研究和发展 (参见, 例如, [18, 83]).

定义 67. 设 A 是伸缩, $0<q\le p<\infty$. 各向异性 *Morrey* 空间$M_{q,A}^p(\mathbb{R}^n)$ 定义为 $\mathbb{R}^n$ 上的所有可测函数 f, 满足

$$\|f\|_{M_{q,A}^p(\mathbb{R}^n)} := \sup_{B\in\mathcal{B}}\left[|B|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(B)}\right] < \infty$$

的全体, 其中 $\mathcal{B}$ 与 (3) 中一致.

容易证明, $M_{q,A}^p(\mathbb{R}^n)$ 是球拟 Banach 函数空间. 由此和 [97, Remark 8.4], 得到 $M_{q,A}^p(\mathbb{R}^n)$ 满足假设 10 和 12, 其中 $X := M_{q,A}^p(\mathbb{R}^n)$, $p_-\in(0,q]$, $\theta_0\in(0,\underline{p})$, $p_0\in(p,\infty)$, $\underline{p} := \min\{p_-,1\}$. 在接下来的内容中, 本文总是令 $HM_{q,A}^p(\mathbb{R}^n)$ 表示各向异性 *Hardy–Morrey* 空间, 它的定义与定义 14 中一致, 取 $X := M_{q,A}^p(\mathbb{R}^n)$. 应用定理 33、34 和 35, 可以得到了 $HM_{q,A}^p(\mathbb{R}^n)$ 的各向异性 Lusin 面积函数、各向异性 Littlewood–Paley g 函数和各向异性 Littlewood–Paley g_λ^* 函数刻画.

ch-Thsm **定理 68.** 设 A 是伸缩, $0<q\le p<\infty$. 则定理 33、34 和 35 对 $X := M_{q,A}^p(\mathbb{R}^n)$ 和 $\lambda\in(2/\min\{1,q\},\infty)$ 成立.

ch-3.23.x1 **注记 69.** (i) 定理 68 是全新的.

(ii) 定理 27、56、58 和 62 以及推论 28 不能应用于各向异性 Morrey 空间 $M_{q,A}^p(\mathbb{R}^n)$, 因为 $M_{q,A}^p(\mathbb{R}^n)$ 的范数缺乏凹性.

此外, $M_{q,A}^p(\mathbb{R}^n)$ 可能不是 q_0-凹的. 实际上, 设 $A := 2I_{n\times n}$, $0<q<p<\infty$, $q_0\in(p,1]$. 假设 $\{f_k\}_{k=1}^\infty$ 是 [41, (2.4)] 中一致特征函数序列, 取 $q := \frac{q}{q_0}$, $p := \frac{p}{q_0}$; 取函数 Φ 与 [41, Theorem 2.15] 中一致, 可得

$$\left\|\sum_{k=1}^\infty f_k\right\|_{M_{\frac{q}{q_0}}^{\frac{p}{q_0}}(\mathbb{R}^n)} = \|\Phi(a)\|_{M_{\frac{q}{q_0}}^{\frac{p}{q_0}}(\mathbb{R}^n)} \sim \|a\|_{l^\infty} = 1 < \sum_{k=1}^\infty \|f_k\|_{M_{\frac{q}{q_0}}^{\frac{p}{q_0}}(\mathbb{R}^n)} = \infty,$$

其中 $a := (1,\dots)\in l^\infty$. 这说明, $M_q^p(\mathbb{R}^n)$ 不是 q_0-凹的. 因此, 定理 45、53 和 54 不能应用于 Morrey 空间, 这些定理由于其依赖于范数的凹性而存在一定的局限性.

然而, 当 $A := 2I_{n\times n}$, $0<q\le p\le 1$, $X := M_{q,A}^p(\mathbb{R}^n)$ 时, 定理 45、53 和 54 分别由 de Almeida 和 Tiago 在 [28, Theorem 3.3、Remark 3.4 和 Proposition 3.8] 中得出, 他们通过使用不同于引理 48 的原子特征来避免对 $\|\cdot\|_{M_q^p(\mathbb{R}^n)}$ 凹性的依赖. 但是, 对于一般的伸缩 A, 目前尚不清楚.

ch-s6-appl2 **Orlicz-Slice Spaces**

最近, Zhang 等人 [106] 最初在 $\mathbb{R}^n$ 上引入了 Orlicz-slice 空间, 它推广了 [2] 中的 slice 空间和 [29] 中的 Wiener-amalgam 空间. 他们还引入了 Orlicz-slice (局部) Hardy 空间, 并在 [105, 106] 中对这些空间进行了完整的实变理论研究. 关于 Orlicz-slice 空间的更多研究, 读者可参见 [44, 45].

函数 $\Phi : [0,\infty) \to [0,\infty)$ 被称为 *Orlicz 函数*, 若它是非递减的; $\Phi(0) = 0$; 对任意的 $t \in (0,\infty)$, $\Phi(t) > 0$; 并且 $\lim_{t\to\infty} \Phi(t) = \infty$. 若存在正常数 C, 对于任意的 $s \in [1,\infty)$(或 $s \in [0,1]$) 和 $t \in [0,\infty)$, $\Phi(st) \le Cs^p\Phi(t)$, 则称函数 Φ 具有*上 (或下) 型为 p*. *Orlicz 空间* $L^\Phi(\mathbb{R}^n)$ 定义为所有在 $\mathbb{R}^n$ 上可测函数 f 满足

$$\|f\|_{L^\Phi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\} < \infty$$

的全体.

定义 70. 设 A 是伸缩, $\ell \in \mathbb{Z}$, $q \in (0,\infty)$, Φ 是 Orlicz 函数. *各向异性 Orlicz-slice 空间*$(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ 定义为 $\mathbb{R}^n$ 上所有可测函数 f 满足

$$\|f\|_{(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} \left[\frac{\|f\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}} < \infty$$

的全体, 其中 B_ℓ 与(2)中一致.

设 A 是伸缩, $\ell \in \mathbb{Z}$, $q \in (0,\infty)$, Φ 是 Orlicz 函数, 具有正下型 p_Φ^- 和正上型 p_Φ^+. 通过类似 [106, Lemmas 2.28 和 4.5] 中使用的论证, 可以发现 $(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ 是球拟 Banach 函数空间, 并且具有绝对连续拟范数. 由此和 [97, Remark 8.14], 能够得出 $(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ 满足假设 10 和 12, 其中 $X := (E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$, $p_- \in (0, \min\{p_\Phi^-, q\}]$, $\theta_0 \in (0, \underline{p})$, $p_0 \in (\max\{p_\Phi^+, q\}, \infty)$, $\underline{p} := \min\{p_-, 1\}$. 在接下来的内容中, 本文总是令 $(HE^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ 表示 *各向异性 Orlicz-slice Hardy 空间*, 它的定义与定义 14 中一致, 取 $X := (E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$.

此外, 通过定理 27、33、34、35、56、58 和 62 以及推论 28, 将 X 取为 $(E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$, 可以得到以下结论.

ch-thorliczslice **定理 71.** 设 A 是伸缩, $\ell \in \mathbb{Z}$, $q \in (0,\infty)$, Φ 是具有正下型 p_Φ^- 的 Orlicz 函数. 则

(i) 定理 27、56、58 和 62 以及推论 28 对 $X := (E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ 结论成立;

(ii) 定理 33、34和 35 对 $X := (E^q_\Phi)_{\ell,A}(\mathbb{R}^n)$ 和 $\lambda \in (\frac{2}{\min\{1, p_\Phi^-, q\}}, \infty)$ 结论成立.

ch-3.23.x2 **注记 72.** 需要指出定理 71是全新的.

设 $\ell\in\mathbb{Z}$, $q\in(0,1)$, Φ 是具有正下型 p_Φ^- 和正上型 p_Φ^+, 满足 $0<p_\Phi^-\le p_\Phi^+<1$ 的 Orlicz 函数, 并且

$$N\in\mathbb{N}\cap\left[\left\lfloor\left(\frac{1}{\min\{p_\Phi^-,q\}}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor+2,\infty\right).$$

通过 [97, Remark 8.14], 可得 $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ 满足定义 14 的所有假设, 其中 $X:=(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$, $p_-\in(0,\min\{p_\Phi^-,q\}]$, $\theta_0\in(0,p_-)$, $p_0\in(\max\{p_\Phi^+,q\},\infty)$. 此外, 取 $q_0=1$. 一方面, 由 [106, Lemma 5.4], 可得, 对于任意非负可测函数列 $\{f_k\}_{k=1}^\infty$ 有

$$\sum_{k=1}^\infty\|f_k\|_{[(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)]^{\frac{1}{q_0}}}\le\left\|\sum_{k=1}^\infty f_k\right\|_{[(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)]^{\frac{1}{q_0}}}.$$

对于任意的 $B\in\mathcal{B}$ 有

ch-eqorliczs (38)
$$\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)}\gtrsim\min\left\{|B|,|B|^{\frac{1}{\theta_0}}\right\}.$$

实际上, 对于任意 $|B|\ge|B_\ell|$ 的 $B\in\mathcal{B}$ 有

ch-eqos1 (39)
$$\begin{aligned}\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)}&=\left\{\int_{\mathbb{R}^n}\left[\frac{\|\mathbf{1}_B\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}}\\&\gtrsim\left(\int_B 1\,dx\right)^{\frac{1}{q}}=|B|^{\frac{1}{q}}.\end{aligned}$$

另一方面, 对于任意 $x_0\in\mathbb{R}^n$, $k\in\mathbb{Z}$ 且 $|B_k|\le|B_\ell|$, $x\in B(x_0,\lambda_-^\ell)$, 以及 $\eta\in(0,p_\Phi^-)$, 由 [99, Remark 4.21(iv)], 可得 $L^\Phi(\mathbb{R}^n)$ 满足假设 10, 其中 $X:=L^\Phi(\mathbb{R}^n)$, $u:=1/\eta$, $p:=\eta$. 因此可得

ch-6.16.1 (40)
$$\|\mathbf{1}_{x_0+B_k}\|_{L^\Phi(\mathbb{R}^n)}^\eta\gtrsim\|[\mathcal{M}(\mathbf{1}_{x_0+B_k})]^{1/\eta}\|_{L^\Phi(\mathbb{R}^n)}^\eta.$$

对于任意 $y\in x+B_\ell$ 有

ch-6.16.2 (41)
$$\mathcal{M}(\mathbf{1}_{x_0+B_k})(y)\ge\frac{1}{|B_\ell|}\int_{x+B_\ell}\mathbf{1}_{x_0+B_k}(z)\,dz\gtrsim\frac{|B_k|}{|B_\ell|}.$$

结合(40)和(41)进一步得

$$\|\mathbf{1}_{x_0+B_k}\|_{L^\Phi(\mathbb{R}^n)}^\eta\gtrsim\left\|\left[\frac{|B_k|}{|B_\ell|}(\mathbf{1}_{x+B_\ell})\right]^{1/\eta}\right\|_{L^\Phi(\mathbb{R}^n)}^\eta\gtrsim|B_k|\,\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}^\eta.$$

因此可得

ch-eqos2 (42)
$$\|\mathbf{1}_{x_0+B_k}\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)}=\left\{\int_{\mathbb{R}^n}\left[\frac{\|\mathbf{1}_{x_0+B_k}\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}}$$

$$\gtrsim \left\{\int_{B(x_0,\lambda_-^\ell)}\left[\frac{\|\mathbf{1}_{x_0+B_k}\|_{L^\Phi(\mathbb{R}^n)}}{\|\mathbf{1}_{x+B_\ell}\|_{L^\Phi(\mathbb{R}^n)}}\right]^q dx\right\}^{\frac{1}{q}}$$

$$\gtrsim |B_k|^{1/\eta}\left\{\int_{B(x_0,\lambda_-^\ell)} 1\,dx\right\}^{\frac{1}{q}} \sim |B_k|^{1/\eta}.$$

通过(39)和(42), 本文发现, 对于任意的 $B\in\mathcal{B}$ 有

$$\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} \gtrsim |B|^{1/q} \gtrsim |B| \text{ 若 } |B|\ge|B_\ell|$$

和

$$\|\mathbf{1}_B\|_{(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)} \gtrsim |B|^{1/p_\Phi^-} \gtrsim |B|^{1/\theta_0} \text{ 若 } |B|\le|B_\ell|.$$

这得到了(38).

因此, 定理 45、53 和 54 的所有假设, 对 $X:=(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ 都成立. 应用定理 45、53 和 54, 可得以下结论.

ch-thorliczslice2 **定理 73.** 设 $\ell\in\mathbb{Z}$, $q\in(0,1)$, 并且 Φ 是一个具有下型 p_Φ^- 和上型 p_Φ^+ 满足 $0<p_\Phi^-\le p_\Phi^+<1$ 的 Orlicz 函数. 则定理 27、56 和 58 在 X 取为 $(E_\Phi^q)_{\ell,A}(\mathbb{R}^n)$ 时结论成立.

ch-rerliczslice2 **注记 74.** 需要指出即使是 $A:=2I_{n\times n}$ 时, 定理 73 也是全新的.

ch-s6-appl3 **Lorentz 空间**

设 $p\in(0,\infty]$ 和 $q\in(0,\infty]$. *Lorentz 空间* $L^{p,q}(\mathbb{R}^n)$ 定义为在 $\mathbb{R}^n$ 上的可测函数 f 并具有以下有限拟范数

ch-6.13.x1 (43)
$$\|f\|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases}\left[\dfrac{q}{p}\displaystyle\int_0^\infty\left\{t^{\frac{1}{p}}f^*(t)\right\}^q\frac{dt}{t}\right]^{\frac{1}{q}} & \text{if } q\in(0,\infty),\\ \sup\limits_{t\in(0,\infty)}\left[t^{\frac{1}{p}}f^*(t)\right] & \text{if } q=\infty\end{cases}$$

的全体, 其中 f^* 表示 f 的*非递增重排*, 即对于任意 $t\in(0,\infty)$, 令

$$f^*(t) := \{\alpha\in(0,\infty):\ d_f(\alpha)\le t\},$$

其中 $d_f(\alpha):=|\{x\in\mathbb{R}^n:\ |f(x)|>\alpha\}|$ 对于任意 $\alpha\in(0,\infty)$.

由 [99, Remarks 2.7(ii), 4.21(ii), and 6.8(iv)], 可得 $L^{p,q}(\mathbb{R}^n)$ 满足定义 14 的所有假设其中 $X:=L^{p,q}(\mathbb{R}^n)$, $p_-\in(0,p]$, $\theta_0\in(0,\underline{p})$, 和 $p_0\in(p,\infty)$, $\underline{p}:=\min\{p_-,1\}$, 并且具有绝对连续拟范数. 接下来, 本文总是令 $H_A^{p,q}(\mathbb{R}^n)$ 为*各向异性 Hardy-Lorentz 空间*, 它的定义与定义 14 中一致, 取 $X:=L^{p,q}(\mathbb{R}^n)$. 通过定理 27、33、34、35、56、58 和 62 以及推论 28, X 被替换为 $L^{p,q}(\mathbb{R}^n)$, 可以得到以下结论.

-thlorentz **定理 75.** 设 A 是伸缩, $p\in(0,\infty)$, 且 $q\in(0,\infty]$. 则

(i) 定理 27、56、58 和 62 以及推论 28在 $X:=L^{p,q}(\mathbb{R}^n)$ 时成立;

(ii) 定理 33、34和 35 在 $X:=L^{p,q}(\mathbb{R}^n)$ 和 $\lambda\in(2/\min\{1,p\},\infty)$ 时成立.

将 $\mathcal{P}(\mathbb{R}^n)$ 表示为 $\mathbb{R}^n$ 上的所有可测函数$p(\cdot)$ 满足

ch-s6e1 (44)
$$0<\widetilde{p_-}:=\operatorname*{ess\,inf}_{x\in\mathbb{R}^n}p(x)\le\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(x)=:\widetilde{p_+}<\infty.$$

$C^{\log}(\mathbb{R}^n)$ 定义为满足全局对数 *Hölder* 连续条件的所有函数 $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ 全体, 也就是说, 存在 $C_{\log}(p),C_\infty\in(0,\infty)$ 和 $p_\infty\in\mathbb{R}$ 使得, 对于任意 $x,y\in\mathbb{R}^n$, 有

$$|p(x)-p(y)|\le\frac{C_{\log}(p)}{\log\left(e+1/|x-y|\right)}$$

和

$$|p(x)-p_\infty|\le\frac{C_\infty}{\log(e+\rho(x))}.$$

ch-3.23.x3 **注记 76.** 设 $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ 和 $q\in(0,\infty)$. 需要指出定理 75(i) 是 [71, Theorems 1 and 2] 的一个特例, 在其中 $p(\cdot)\equiv p\in(0,\infty)$, 而定理 75(ii) 改进了 [76, Theorems 2.7, 2.8 and 2.9] 中的对应的结果, 将 $p\in(0,1]$ 的范围扩大到 $p\in(0,\infty)$. 虽然变指标 Hardy–Lorentz 空间 $L^{p(\cdot),q}(\mathbb{R}^n)$ 也是球拟 Banach 函数空间, 但是 [71, Theorems 1 and 2] 无法由定理 27 和 62 推导出来. 这是因为 Hardy–Littlewood 极大算子在 $L^{p(\cdot),q}(\mathbb{R}^n)$ 的相关空间上的有界性仍然未知, 使得无法验证假设 12 在 $X:=L^{p(\cdot),q}(\mathbb{R}^n)$ 的情形.

此外, 取 $q_0\in(\max\{p,q\},1]$. 从 [55, Theorem 6(iii)] 和 [40, Propostion 1.4.5(13)] 知, 存在正常数 C 使得, 对于任意非负可测函数列 $\{f_k\}_{k=1}^\infty$ 有

$$\begin{aligned}\left[\sum_{k=1}^\infty\|f_k\|_{L^{\frac{p}{q_0},\frac{q}{q_0}}(\mathbb{R}^n)}\right]^{\frac{q}{q_0}}&=\left[\sum_{k=1}^\infty\|f_k\|_{\Lambda_{\frac{q}{q_0},\frac{q}{p}t^{\frac{q}{p}-1}}}\right]^{\frac{q}{q_0}}=\left[\sum_{k=1}^\infty\left\|f_k^{\frac{q}{q_0}}\right\|_{\Lambda_{1,\frac{q}{p}t^{\frac{q}{p}-1}}}^{\frac{q_0}{q}}\right]^{\frac{q}{q_0}}\\&\le C\left\|\left(\sum_{k=1}^\infty f_k\right)^{\frac{q}{q_0}}\right\|_{\Lambda_{1,\frac{q}{p}t^{\frac{q}{p}-1}}}=C\left\|\sum_{k=1}^\infty f_k\right\|_{\Lambda_{\frac{q}{q_0},\frac{q}{p}t^{\frac{q}{p}-1}}}^{\frac{q}{q_0}}\\&=C\left\|\sum_{k=1}^\infty f_k\right\|_{L^{\frac{p}{q_0},\frac{q}{q_0}}(\mathbb{R}^n)}^{\frac{q}{q_0}}\end{aligned}$$

其中 Λ 与 [55, p. 270] 的 6-7 行一致, 这进一步意味着

$$\sum_{k=1}^\infty\|f_k\|_{L^{\frac{p}{q_0},\frac{q}{q_0}}(\mathbb{R}^n)}\le C\left\|\sum_{k=1}^\infty f_k\right\|_{L^{\frac{p}{q_0},\frac{q}{q_0}}(\mathbb{R}^n)}.$$

此外, 由(43), 对于任意 $B\in\mathcal{B}$ 有

$$\|\mathbf{1}_B\|_{L^{p,q}(\mathbb{R}^n)}=\left\{\frac{q}{p}\int_0^{|B|}t^{\frac{q}{p}-1}dt\right\}^{\frac{1}{q}}=|B|^{\frac{1}{p}}\geq\min\left\{|B|^{\frac{1}{q_0}},|B|^{\frac{1}{\theta_0}}\right\}.$$

因此, 所有定理 45、53 和 54 的假设都满足, 其中 $X:=L^{p,q}(\mathbb{R}^n)$. 应用定理 45、53 和 54, 可以得到以下结论.

ch-thlorentz2 **定理 77.** 若 $p\in(0,1)$ 且 $q\in(0,1)$, 则定理 45、53 和 54 中将 X 取为 $L^{p,q}(\mathbb{R}^n)$ 时成立.

ch-remrentz2 **注记 78.** 需要指出即使 $A:=2I_{n\times n}$, 定理 77 也是全新的.

ch-s6-appl4 变指标 Lebesgue 空间

对于任意 $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, *变指标 Lebesgue 空间* $L^{p(\cdot)}(\mathbb{R}^n)$ 定义为在 $\mathbb{R}^n$ 上所有可测函数 f, 使得

$$\int_{\mathbb{R}^n}|f(x)|^{p(x)}\,dx<\infty$$

的全体, 并赋有*拟范数* $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, 定义如下:

ch-6.8.x1 (45)
$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=\inf\left\{\lambda\in(0,\infty):\ \int_{\mathbb{R}^n}\left[\frac{|f(x)|}{\lambda}\right]^{p(x)}\,dx\leq 1\right\}.$$

设 $p(\cdot)\in C^{\log}(\mathbb{R}^n)$. 由 [99, Remarks 2.7(iv), 4.21(v), and 6.8(vii)], 可得 $L^{p(\cdot)}(\mathbb{R}^n)$ 满足定义 14 的所有假设, 其中 $X:=L^{p(\cdot)}(\mathbb{R}^n)$, $p_-:=\widetilde{p_-}$, $\theta_0\in(0,\underaccent{\sim}{p})$, 和 $p_0\in(\widetilde{p_+},\infty]$, $\widetilde{p_-}$ 和 $\widetilde{p_+}$ 与(44)一致, 并且具有绝对连续拟范数. 接下来, 本文总是令 $H_A^{p(\cdot)}(\mathbb{R}^n)$ 为*各向异性变指标 Hardy 空间*, 它的定义与定义 14 中一致, 取 $X:=L^{p(\cdot)}(\mathbb{R}^n)$. 此外, 由定理 27、33、34、35、56、58 和 62 以及推论 28 将 X 取为 $L^{p(\cdot)}(\mathbb{R}^n)$, 可以得到以下结论.

ch-thvariable **定理 79.** 设 A 是伸缩, $p(\cdot)\in C^{\log}(\mathbb{R}^n)$. 则

(i) 定理 27、56、58 和 62 以及推论 28 取 $X:=L^{p(\cdot)}(\mathbb{R}^n)$ 时成立;

(ii) 定理 33、34 和 35 取 $X:=L^{p(\cdot)}(\mathbb{R}^n)$ 和 $\lambda\in(2/\min\{1,\widetilde{p}_-\},\infty)$ 时成立, 其中 $\widetilde{p_-}$ 与(44)一致.

ch-3.23.x4 **注记 80.** 定理 79(i) 在 [48, Theorems 1, 2, and 3, and Corollary 1] 中得到, 而定理 79(ii) 则改进了 [72, Theorems 6.1, 6.2, and 6.3] 中 $\lambda\in(1+2/\min\{2,\widetilde{p}_-\},\infty)$ 的范围到 $\lambda\in(2/\min\{1,\widetilde{p}_-\},\infty)$.

设 A 是伸缩, $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ 且满足 $0<\widetilde{p_-}\le\widetilde{p_+}<1$,

$$N\in\mathbb{N}\cap\left[\left\lfloor\left(\frac{1}{\widetilde{p_-}}-1\right)\frac{\ln b}{\ln(\lambda_-)}\right\rfloor+2,\infty\right).$$

由 [99, Remarks 2.7(iv) and 4.21(v)] 得 $L^{p(\cdot)}(\mathbb{R}^n)$ 满足定义 14 的所有假设, 其中 $X:=L^{p(\cdot)}(\mathbb{R}^n)$, $p_-\in(0,\widetilde{p_-}]$, $\theta_0\in(0,p_-)$, 和 $p_0\in(\widetilde{p_+},\infty)$. 此外, 取 $q_0\in(\widetilde{p_+},1]$. 一方面, 从 [102, Remark 2.1(iv)] 可得, 对任意非负可测函数列 $\{f_k\}_{k=1}^\infty$ 有

$$\sum_{k=1}^\infty\|f_k\|_{L^{\frac{p(\cdot)}{q_0}}(\mathbb{R}^n)}\le\left\|\sum_{k=1}^\infty f_k\right\|_{L^{\frac{p(\cdot)}{q_0}}(\mathbb{R}^n)}.$$

另一方面, 由(45)可得, 对于任意 $B\in\mathcal{B}$ 有

$$\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\gtrsim\min\left\{|B|^{\frac{1}{\widetilde{p_+}}},|B|^{\frac{1}{\widetilde{p_-}}}\right\}>\min\left\{|B|^{\frac{1}{q_0}},|B|^{\frac{1}{\theta_0}}\right\}.$$

因此, $L^{p(\cdot)}(\mathbb{R}^n)$ 满足定理 45 的所有假设, 其中 $X:=L^{p(\cdot)}(\mathbb{R}^n)$. 在这种情况下, 定理 45、53 和 54 分别在 [67, Theorems 1, 2, and 3] 中得到.

ch-s6-appl5

混合范数 Lebesgue 空间

设 $\vec{p}:=(p_1,\dots,p_n)\in(0,\infty]^n$. 混合范数 *Lebesgue* 空间 $L^{\vec{p}}(\mathbb{R}^n)$ 的定义如下:

ch-6.8.x2

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)}:=\left\{\int_{\mathbb{R}}\cdots\left[\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}|f(x_1,\dots,x_n)|^{p_1}\,dx_1\right\}^{\frac{p_2}{p_1}}\,dx_2\right]^{\frac{p_3}{p_2}}\cdots\,dx_n\right\}^{\frac{1}{p_n}}\tag{46}$$

当某些 $i\in\{1,\dots,n\}$, $p_i=\infty$ 时, 做相应的一般定义修改.

对于任意 $\vec{p}\in(0,\infty]^n$, 由 [104, p. 2047] 和 [47, Lemmas 3.2 and 4.4], 可得 $L^{\vec{p}}(\mathbb{R}^n)$ 满足定义 14 中的所有假设, 其中 $X:=L^{\vec{p}}(\mathbb{R}^n)$, $p_-:=\widehat{p_-}$, $\theta_0\in(0,\underline{\widehat{p}})$, $p_0\in(\theta_0,\infty)$, $\widehat{p_-}:=\min\{p_1,...,p_n\}$ 和 $\underline{\widehat{p}}:=\min\{1,\widehat{p_-}\}$, 并且具有绝对连续拟范数. 接下来, 本文总是令 $H_A^{\vec{p}}(\mathbb{R}^n)$ 为各向异性混合范数 *Hardy* 空间, 它的定义与定义 14 中一致, 取 $X:=L^{\vec{p}}(\mathbb{R}^n)$. 此外, 通过将定理 27、33、34、35、56、58 和 62 以及推论 28 中 X 取为 $L^{\vec{p}}(\mathbb{R}^n)$, 可以得到以下结论.

ch-thmix

定理 81. 设 A 是伸缩, $\vec{p}\in(0,\infty)^n$. 则

(i) 定理 27、56、58 和 62 以及推论 28 在 $X:=L^{\vec{p}}(\mathbb{R}^n)$ 时成立;

(ii) 定理 33、34 和 35 在 $X:=L^{\vec{p}}(\mathbb{R}^n)$ 和 $\lambda\in(2/\min\{1,\widehat{p_-}\},\infty)$ 时成立, 其中 $\widehat{p_-}:=\min\{p_1,...,p_n\}$.

ch-3.23.x5 **注记 82.** (i) 需要指出定理 81(i) 在 [49, Theorems 3.4, 4.1, and 5.3 and Corollary 3.9] 中得到, 而定理 81(ii) 改进了 [47, Theorems 6.2, 6.3, and 6.4] 中 $\lambda \in (1 + 2/\min\{2, \widehat{p_-}\}, \infty)$ 的范围到 $\lambda \in (2/\min\{1, \widehat{p_-}\}, \infty)$.

(ii) 设 $\vec{a} := (a_1, \ldots, a_n) \in [1, \infty]^n$. 当定理 81(i) 取

$$A := \begin{pmatrix} 2^{a_1} & 0 & \cdots & 0 \\ 0 & 2^{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2^{a_n} \end{pmatrix}$$

时, 完全回答了 [20] 中提出的关于 $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ 的对偶空间的公开问题.

设 $\vec{p} \in (0,1)^n$, 取 $q_0 \in (\widehat{p_+}, 1]$. 从 (46) 和 [68, (9)], 得到对于任意非负可测函数列 $\{f_k\}_{k=1}^{\infty}$ 和任意 $B \in \mathcal{B}$, 有

$$\sum_{k=1}^{\infty} \|f_k\|_{L^{\frac{\vec{p}}{q_0}}(\mathbb{R}^n)} \le C \left\| \sum_{k=1}^{\infty} f_k \right\|_{L^{\frac{\vec{p}}{q_0}}(\mathbb{R}^n)}$$

和

$$\|\mathbf{1}_B\|_{L^{\vec{p}}(\mathbb{R}^n)} \gtrsim \min\left\{ |B|^{\frac{1}{\widehat{p_+}}}, |B|^{\frac{1}{\widehat{p_-}}} \right\} = \min\left\{ |B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}} \right\}.$$

因此, $L^{\vec{p}}(\mathbb{R}^n)$ 满足定理 45 的所有假设, 其中 $X := L^{\vec{p}}(\mathbb{R}^n)$. 在这种情况下, 定理 45、53 和 54 分别在 [68, Theorems 3.1, 4.1, and 4.3] 中得到.

ch-s6-appl6 加权 Lebesgue 空间

设 $p \in (0, \infty]$ 和 $w \in \mathcal{A}_\infty(A)$. 从 [99, Remarks 2.7(iii), 4.21(iii), and 6.8(v)] 中得到 $L_w^p(\mathbb{R}^n)$ 满足定义 14 中的所有假设, 其中 $X := L_w^p(\mathbb{R}^n)$, $p_- \in (0, p/q_w]$, $\theta_0 \in (0, \min\{1, p_-\})$ 和 $p \in (\theta_0, \infty)$, q_w 与 (22) 一致, 并且具有绝对连续拟范数. 接下来, 本文总是令 $H_w^p(\mathbb{R}^n)$ 为*各向异性加权 Hardy 空间*, 它的定义与定义 14 中一致, 取 $X := L_w^p(\mathbb{R}^n)$. 将 X 取为 $L_w^p(\mathbb{R}^n)$, 应用定理 27、33、34、35、56、58 和 62 以及推论 28 可以得到以下结论.

ch-thwei **定理 83.** 设 A 是伸缩, $p \in (0, \infty)$, 且 $w \in \mathcal{A}_\infty(A)$. 则

(i) 定理 27、56、58 和 62 以及推论 28 在 $X := L_w^p(\mathbb{R}^n)$ 时成立;

(ii) 定理 33、34 和 35 在 $X := L^p_w(\mathbb{R}^n)$ 和 $\lambda \in (2/\min\{1, q_w/p\}, \infty)$ 成立, 其中 q_w 与 (22) 一致.

ch-3.27.x1 **注记 84.** 需要指出定理 83(i) 是全新的, 定理 83(ii) 改进了 [58, Theorems 2.14, 3.1, and 3.9] 中相应的结果, 将 $p \in (0, 1]$ 的范围扩展到 $p \in (0, \infty)$. 然而, 由于当 $X := L^p_w(\mathbb{R}^n)$ 时, (24) 可能不成立, 因此定理 45、53 和 54 无法应用于加权 Lebesgue 空间.

h-s6-appl7

Orlicz 空间

设 Φ 是一个 Orlicz 函数, 具有正下型 p_Φ^- 和正上型 p_Φ^+. 从 [99, Remarks 2.7(iii), 4.21(iv), and 6.8(vi)] 中, 得到 $L^\Phi(\mathbb{R}^n)$ 满足定义 14 中的所有假设, 其中 $X := L^\Phi(\mathbb{R}^n)$, $p_- \in (0, p_\Phi^-]$, $\theta_0 \in (0, \min\{p_\Phi^-, 1\})$, $p_0 \in (\max\{p_\Phi^+, 1\}, \infty)$, 并且具有绝对连续拟范数. 接下来, 本文总是令 $H^\Phi_A(\mathbb{R}^n)$ 为**各向异性** *Orlicz–Hardy* **空间**, 它的定义与定义 14 中一致, 取 $X := L^\Phi(\mathbb{R}^n)$. 通过将定理 27、33、34、35、56、58 和 62 以及推论 28 中将 X 取为 $L^\Phi(\mathbb{R}^n)$, 可以得到以下结论.

h-thorlicz **定理 85.** 设 A 是伸缩, Φ 是一个 Orlicz 函数, 具有下型 $p_\Phi^- \in (0, \infty)$. 则

(i) 定理 27、56、58 和 62 以及推论 28 在 $X := L^\Phi(\mathbb{R}^n)$ 时成立;

(ii) 定理 33、34 和 35 在 $X := L^\Phi(\mathbb{R}^n)$ 和 $\lambda \in (2/\min\{1, p_\Phi^-\}, \infty)$ 时成立.

ch-3.27.x2 **注记 86.** 需要指出定理 85(i) 是全新的, 定理 85(ii) 改进了 [58, Theorems 2.14, 3.1, and 3.9] 中相应的结果, 将 $p_\Phi^- \in (0, 1]$ 的范围扩展到 $p_\Phi^- \in (0, \infty)$.

此外, 取 $q_0 \in (p_\Phi^+, 1]$. 则由 [106, Remarks 5.3] 和 [46, (25)], 对于任意非负可测函数列 $\{f_k\}_{k=1}^\infty$ 和任意 $B \in \mathcal{B}$, 有

$$\sum_{k=1}^\infty \|f_k\|_{[L^\Phi(\mathbb{R}^n)]^{\frac{1}{q_0}}} \le \left\| \sum_{k=1}^\infty f_k \right\|_{[L^\Phi(\mathbb{R}^n)]^{\frac{1}{q_0}}}$$

和

$$\|\mathbf{1}_B\|_{L^\Phi(\mathbb{R}^n)} \gtrsim \min\left\{|B|^{\frac{1}{p_\Phi^-}}, |B|^{\frac{1}{p_\Phi^+}}\right\} \ge \min\left\{|B|^{\frac{1}{q_0}}, |B|^{\frac{1}{\theta_0}}\right\}.$$

因此, 所有定理 45、53 和 54 的假设都满足, 其中 $X := L^\Phi(\mathbb{R}^n)$. 应用定理 45、53 和 54, 可以得到以下结论.

-thorlicz2 **定理 87.** 设 Φ 是一个 Orlicz 函数, 其下型 p_Φ^- 和上型 p_Φ^+ 满足 $0 < p_\Phi^- \le p_\Phi^+ < 1$. 则定理 45、53 和 54 在 X 取 $L^\Phi(\mathbb{R}^n)$ 时成立.

h-rerlicz2 **注记 88.** 需要指出定理 87 即使当 $A := 2I_{n\times n}$ 时也是全新的.

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